# New QM framework

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**Abstract**

I propose a model wherein a system is represented by a finite sequence of natural numbers. These numbers are thought of as population numbers in statistical ensemble formed as a sample with replacement of entities (microstates) from some abstract set. I derive the concepts of energy and of temperature. I show the analogy between energy spectra computed from the model and energy spectra of some known constructs, such as particle in a box and quantum harmonic oscillator. The presented model replaces the concept of wave function with *knowledge vector*. I derive Schrödinger-type equation for knowledge vector and discuss principal differences with Schrödinger's equation. The model retains major QM hallmarks such as wave-particle duality, violation of Bell's inequalities, quantum Zeno effect, while avoiding controversial concept of wave function collapse. Unlike QM and Newtonian mechanics the presented model has the Second Law of Thermodynamics built-in; in particular, it is not invariant with respect to time reversal.

> *As for prophecies, they will pass away; as for tongues, they will cease; as for knowledge, it will pass away.*
> 1 Corinthians 13:8

## 1. PREAMBLE

Physical properties, such as temperature, energy, entropy, pressure, and phenomena such as Bose-Einstein condensation (BEC) are exhibited not just by "real" physical systems, but also by virtual entities such as binary or character strings (Wislicki, 1990; Frenzen, 1993; Viznyuk, 2010), world wide web (Albert & Barabási, 2002), business and citation networks (Bianconi & Barabási, 2000; Barabási & Bonabeau, 2003), economy (Bouchaud, 2000; Garrett, 2013; Sieniutycz & Salamon, 1990). There must be an underlying mechanism which accounts for the *grand commonality* in observed behavior of vastly different entities. Quoting (Barabási & Bonabeau, 2003): *Scientists have recently discovered that various complex systems have an underlying architecture governed by shared organizing principles*. Earlier it was argued *that present-day quantum mechanics is a limiting case of some more unified scheme… Such a theory would have to provide, as an appropriate limit, something equivalent to a unitarily evolving state vector* $|\psi\rangle$ (Penrose, 1996).

There are two factors present in all theories. One is the all-pervading *time*, and the other is the observer's *mind*. A successful *grand commonality* model must explain the nature of time, specify mechanism of how the *physical reality* projects onto the mind of observer, and relate time to that projection.

I strive to provide definition for every notion I use. I call physical reality (i.e. the "real" world) the *underlying system*. The underlying system is represented by its state vector $\boldsymbol{x}$. Observer's mind is the *basis*. *Observables* are the basis vectors. *Knowledge* vector $\boldsymbol{y}$ is the state vector $\boldsymbol{x}$ represented in the *basis*. The concept of knowledge vector may seem similar to wave function in Niels Bohr interpretation according to which the wave function *is not to be taken*

*seriously as describing a quantum-level physical reality, but is to be regarded as merely referring to our (maximal) "knowledge" of a physical system…* (Penrose, 1996).

The state vector has an associated value of *proper time* (Viznyuk, 2011) which serves as an ordering parameter for different states of underlying system. State vector is completely defined by a finite sequence of natural numbers $(n_i)$. I call $(n_i)$ the *population numbers* of microstates $\{i\}$ from set $G$; $\{i\} \in G$. I do not speculate what is microstate or set $G$, leaving them as abstract notions. I think of sequence $(n_i)$ as a *sample with replacement* of microstates $\{i\}$ from set $G$. I call such sample the *statistical ensemble*. Henceforth the notion of *physical reality* is reduced to a sequence of natural numbers $(n_i)$ which do not have to be physicalized in any way.

The *proper time* has been previously defined (Viznyuk, 2011) as the ordering parameter for the states of statistical ensemble:

$$t = \ln N \qquad \text{, where} \qquad N = \sum_{i \in G} n_i \qquad (1)$$

The proper time is quantized with time quantum $\tau = \Delta(\ln N) = 1/N$. I combine this definition of time with the following rule on time increments:

The positive $\Delta t > 0$ direction of time change is when:

$$\Delta N = \Delta N|_{\Delta t > 0} \equiv \sum_{\Delta n_i > 0} \Delta n_i \qquad (2)$$

, where all $\Delta n_i$ are non-negative.

The negative $\Delta t < 0$ direction of time change is when:

$$\Delta N = \Delta N|_{\Delta t < 0} \equiv \sum_{\Delta n_i < 0} \Delta n_i \qquad (3)$$

, where all $\Delta n_i$ are non-positive.

If $(n_i)$ and $(n_i')$ are such that some $\Delta n_i = n_i' - n_i$ are positive and some are negative, then state vectors $(n_i)$ and $(n_i')$ do not connect by timeline. There can be multiple timelines (histories) connecting two state vectors, as well as none. For the given state vector, a choice of observation basis defines *knowledge vector*.

In QM a common approach is to apply Schrödinger's equation in forward-only manner, to find conditional probabilities of future measurement outcomes [eq. (107,109) in Section 4]. What happens if same equation is solved backward in time starting with the present *known* state? If we do, we would find the past too is only defined in terms of conditional probabilities.

In my model any *known fact* from the past is an artifact of the present state. What observer thinks as the *past*, the *present*, or the *future* are represented by the knowledge vector in the *present*. Thus, there is only *present*.

To further develop the model, I derive the notions of energy and of temperature. I derive the equation of motion for knowledge vector and discuss its similarity and differences with Schrödinger equation. I touch upon the notions of ergodicity; open/closed systems; conservation of energy; Bell's inequalities; Haag's theorem; quantum Zeno effect; and the notion of memory.

## 2. ENERGY

The base tenet of the model is that the combination $(n_{i \in G})$ completely defines the underlying system. I call $(n_i)$ combination a *mode*. Since *mode* is formed as a sample with replacement, the [unconditional] probability of finding underlying system in a particular mode is given by multinomial probability mass function:

$$P\big((n_i); \ N, (p_i)\big) = N! \prod_{i \in G} \frac{p_i^{n_i}}{n_i!} \qquad (4)$$

, where $p_i$ is the probability of sampling microstate $i$ from set $G$. Within the context of the model

$$p_i = \frac{1}{M} \quad \forall \; i \in G \tag{5}$$

, where $M$ is the cardinality of set $G$. I introduce functions $\mathcal{E}$, $\mu$, $\chi$ as follows:

$$\ln P\big((n_i); \; N, (p_i)\big) = \mu\big(N, (p_i)\big) - \mathcal{E}\big((n_i); \; N, (p_i)\big) \tag{6}$$

$$\mu\big(N, (p_i)\big) = \ln P\big((n_i \equiv N \cdot p_i); \; N, (p_i)\big) = \ln \Gamma(N+1) - \chi\big(N, (p_i)\big) \tag{7}$$

$$\chi\big(N, (p_i)\big) = \sum_{i \in G} [\ln \Gamma(Np_i + 1) - Np_i \ln p_i] \tag{8}$$

$$\mathcal{E}\big((n_i); \; N, (p_i)\big) = \mu\big(N, (p_i)\big) - \ln P\big((n_i); \; N, (p_i)\big) = \\ \sum_{i \in G} \left[ \ln \frac{\Gamma(n_i + 1)}{\Gamma(Np_i + 1)} + (Np_i - n_i) \cdot \ln p_i \right] \tag{9}$$

$$\mathcal{E}\big((n_i); \; N, (p_i)\big) \geq 0; \qquad \mathcal{E}\big((n_i = Np_i \; \forall \; i \in G); \; N, (p_i)\big) = 0 \tag{10}$$

, where $\Gamma(x)$ is *gamma* function. With (7-9), I rewrite (4) as

$$P\big((n_i); \; N, (p_i)\big) = exp\left(\mu\big(N, (p_i)\big) - \mathcal{E}\big((n_i); N, (p_i)\big)\right) \tag{11}$$

From (11) the probability of observing statistical ensemble of $N$ microstates in a particular *mode* is determined solely by the value of $\mathcal{E}\big((n_i); N, (p_i)\big)$. If I'm to use $\mathcal{E}$ as a single independent variable, I can write the probability mass function in $\mathcal{E}$ domain as:

$$P\big(\mathcal{E}; N, (p_i)\big) = g\big(\mathcal{E}; N, (p_i)\big) \cdot exp\big(\mu\big(N, (p_i)\big) - \mathcal{E}\big) \tag{12}$$

Here $g\big(\mathcal{E}; N, (p_i)\big)$ is the multiplicity (degeneracy) of the given $\mathcal{E}$ value[1], i.e. a number of ways the same value of $\mathcal{E}$ is realized by different modes with given parameters $N, (p_i)$. There is no analytic expression for $g\big(\mathcal{E}; N, (p_i)\big)$, however, it is numerically computable. Table 1 contains $\mathcal{E}$, $g\big(\mathcal{E}; N, (p_i)\big)$ values calculated for several sets of parameters $N, (p_i)$. Figures 1-2 show distinct values of $\mathcal{E}$ in increasing order for several values of parameter $N$ and probabilities (5) calculated from (9), using (Yamanaka, Kawano, & Y., 2007) algorithm for finding partitions $(n_i)$ of integer $N$ into $\leq M$ parts (Viznyuk, OEIS sequence A210237, 2012). The sum of $g\big(\mathcal{E}; N, (p_i)\big)$ over all distinct values of $\mathcal{E}$ is the total number of modes. It is equal to the number of ways to distribute $N$ indistinguishable balls into $M$ distinguishable cells:

$$L(N, M) = \sum_{\{\mathcal{E}\}} g\big(\mathcal{E}; N, (p_i)\big) = \frac{(N + M - 1)!}{N! (M - 1)!} \tag{13}$$

, where sum is over all distinct values of $\mathcal{E}$. Figure 3 shows the total number $L(N, M)$ of distinguishable states of statistical ensemble, and the total number of distinct values $\{\mathcal{E}\}$ as functions of $N$ for two sets of probabilities (5), calculated from (13) and (9) using (Yamanaka, Kawano, & Y., 2007) algorithm. The graphs demonstrate the following:

- For probabilities (5) the average degeneracy of $\{\mathcal{E}\}$ levels approaches $M!$ as $N \to \infty$.

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[1] For a case of statistical ensemble with microstate probabilities (5); the multiplicity of $\mathcal{E}$ is the multiplicity of the value of multinomial coefficient in (4) (Viznyuk, OEIS sequence A210238, 2012)

This statement can be expressed as:

$$M! \cdot \lim_{N \to \infty} \sum_{\{\mathcal{E}\}} 1 = \frac{(N + M - 1)!}{N! \, (M - 1)!} \tag{14}$$

Here $\sum_{\{\mathcal{E}\}} 1$ sum represents the number of distinct values of $\mathcal{E}$ for the given parameters $N, M$. As $g(\mathcal{E}; N, (p_i))$ is not a smooth function of $\mathcal{E}$ (see Table 1), there could be no true probability density in $\mathcal{E}$ domain. However, I shall derive pseudo probability density to be used in expressions involving integration by $\mathcal{E}$ in *thermodynamic limit*. To be able to use analytical math I have to extend (7-11) from discrete variables $(n_i)$ to continuous domain. I call

- *Thermodynamic limit* is the approximation of large population numbers:

$$n_i \gg 1 \ \forall \ i \in G \tag{15}$$

In thermodynamic limit, I shall use Stirling's approximation for factorials

$$\ln n! \approx \frac{1}{2} \ln 2\pi n + n \ln n - n \tag{16}$$

With (5) it allows rewriting of (7-9) as

$$\mu(N, (p_i)) \cong -\frac{1}{2}\left[(M - 1) \cdot \ln 2\pi N + \ln \prod_{i \in G} p_i\right] = \mu(N, M) = \frac{M}{2}\ln M - \frac{M - 1}{2}\ln 2\pi N \tag{17}$$

$$\mathcal{E}((n_i); N, (p_i)) \cong \sum_{i \in G}\left(n_i + \frac{1}{2}\right) \cdot \ln \frac{n_i}{Np_i} = \sum_{i \in G}\left(n_i + \frac{1}{2}\right) \cdot \ln n_i - \left(N + \frac{M}{2}\right)\ln \frac{N}{M} \tag{18}$$

Figure 4 demonstrates function $\mu(N, (p_i))$ calculated for two sets of parameters $(p_i)$ using exact expression (7) and approximate formula (17). In thermodynamic limit, $\mathcal{E}$ is a smooth function of $(n_i)$ approximated by positive semi-definite quadratic form of $\{n_i/N - p_i\}$ in the vicinity of its minimum (10):

$$\mathcal{E} \cong \sum_{\substack{i \in G \\ j \in G}} b_{i,j} \cdot \left(\frac{n_i}{N} - p_i\right) \cdot \left(\frac{n_j}{N} - p_j\right) \tag{19}$$

Knowing the covariance matrix (Forbes, Evans, Hastings, & Peacock, 2010) of multinomial distribution (4) allows reduction of (19) to a diagonal form. The covariance matrix, divided by $N$ is:

$$\sigma_{ij} = \delta_{ij} \cdot p_j - p_i \cdot p_j \qquad \text{, where} \qquad \delta_{i=j} = 1 \ ; \ \delta_{i \neq j} = 0 \tag{20}$$

The rank of $\sigma_{ij}$ is $M - 1$. If $d_{ij}$ is a diagonal form of $\sigma_{ij}$, the eigenvalues of $\sigma_{ij}$ are $d_i = d_{ii}$:

$$d_{ij} = diag(\sigma_{ij}) \ ; \quad d_i = d_{ii} \quad ; \quad d_1 \equiv 0 \quad ; \quad d_{i>1} > 0 \tag{21}$$

For equal probabilities (5) $d_{i>1} = 1/M$. I transform to new discrete variables:

$$x_{i>1} = \sum_{j \in G}\left(\frac{n_j}{N} - p_j\right)\frac{\Theta_{ji}}{\sqrt{d_i}} = \frac{\sqrt{M}}{N}\sum_{j \in G}\left(n_j - \frac{N}{M}\right) \cdot \Theta_{ji} \qquad ; \ x_1 \equiv 0 \tag{22}$$

, where $\Theta_{ij}$ is matrix with columns as unit eigenvectors of $\sigma_{ij}$ corresponding to eigenvalues (21). The eigenvector $\Theta_{i1}$ corresponding to eigenvalue $d_1 \equiv 0$ is perpendicular to hyper-plane (1) defined by $\sum_{i \in G} n_i = N$ in $M$-dimensional space of $(n_i)$ coordinates, while vector $(n_i/N - p_i)$ is parallel to the hyper-plane. Therefore, $x_1 \equiv 0$ in (22). Now I can rewrite (19) in terms of new variables $(x_i)$ as

$$\mathcal{E} = \frac{N}{2}\sum_{i \in G} x_i^2 \tag{23}$$

I call $(x_i)$ the *canonical variables* of statistical ensemble, and $\mathbf{x}$ a canonical *state vector*. I call parameter $\mathcal{E}$ the *energy* of statistical ensemble. The canonical variables $(x_i)$ constitute the observable values in the basis of eigenvectors of $\sigma_{ij}$. As stated in preamble, a *basis* is associated

with the given observer, therefore basis vectors may differ from eigenvectors of $\sigma_{ij}$. If a basis is obtained from eigenvectors of $\sigma_{ij}$ via an orthogonal transformation, the quadratic form (23) is preserved. Hence, I state the *conservation of energy law* as follows: *the energy of the system is conserved under orthogonal transformations of the basis*. In layman's terms it means the energy may change from one form to another (e.g. from potential energy to kinetic) while the total energy of the system is conserved. The conservation of energy law in this form differs from the common one (Conservation of Energy, Wikipedia) which "… *states that the total [energy](#) of an [isolated system](#) remains … conserved over time*". Time model outlined in (Viznyuk, 2011) leads to the following time evolution law for $(x_i)$ and $\mathcal{E}$ in thermodynamic limit:

$$x_i = x_{0_i} \cdot exp(t_0 - t) \qquad\qquad \mathcal{E} = \mathcal{E}_0 \cdot exp(t_0 - t) \qquad (24)$$

Therefore, in presented framework, the total energy of the system *is not conserved over time*. Commonly, the exponential decay (24) has much longer characteristic timescale than orthogonal transformations of the observation basis, as I discuss in Section 4. Therefore, the common formulation of conservation of energy law holds sway in usual circumstances.

Figure 5 demonstrates function $\sqrt{\mathcal{E}/N}$ calculated for two sets of parameters $(p_i)$ using exact expression (9) and approximations (18), and (23). I plotted $\sqrt{\mathcal{E}/N}$ instead of $\mathcal{E}$ to show asymptotic behavior of (9) and (18) in comparison with quadratic form (23). Using (17) and (23) I obtain multivariate normal approximation (Forbes, Evans, Hastings, & Peacock, 2010) to multinomial distribution (4) as

$$P\big((x_i);\ N, (p_i)\big) \cong (2\pi N)^{\frac{1-M}{2}} \cdot exp\left[-\frac{1}{2}\sum_{i \in G}(Nx_i^2 + \ln p_i)\right] = exp\left[\mu\big(N, (p_i)\big) - \frac{N}{2}\sum_{i \in G} x_i^2\right] \quad (25)$$

Figure 6 shows graphs of $\ln P\big((n_i);\ N, (p_i)\big)$ as a function of $n_1$ calculated for $N = 1000$ and four sets of probabilities $(p_i)$, using exact formula (4), and multivariate normal approximation (25). In order to derive pseudo probability density in $\mathcal{E}$ domain, I note that:

- In thermodynamic limit the number $L(\mathcal{E}_0; N, M)$ of distinguishable states of statistical ensemble having $\mathcal{E} \le \mathcal{E}_0$ is proportional to the volume of $(M - 1)$ –dimensional sphere (26) of radius $\sqrt{N\mathcal{E}_0}$. This statement can be expressed as

$$L(\mathcal{E}_0; N, M) = \lim_{N \to \infty} \sum_{\{\mathcal{E}\} \le \mathcal{E}_0} g(\mathcal{E}; N, M) = a(N, M) \cdot (N\mathcal{E}_0)^{\frac{M-1}{2}} \qquad (27)$$

The sum in (27) is over all distinct values of $\mathcal{E}$ which are less or equal than $\mathcal{E}_0$. The function $a(N, M)$ is determined from normalization requirement:

$$1 = \sum_{\{\mathcal{E}\}} P(\mathcal{E}; N, M) = \sum_{\{\mathcal{E}\}} g(\mathcal{E}; N, M) \cdot exp(\mu(N, M) - \mathcal{E}) \qquad (28)$$

In order to convert from sums to integrals over continuous variable $\mathcal{E}$ I define pseudo density $g(\mathcal{E}; N, M)$ of distinguishable states of statistical ensemble as

$$g(\mathcal{E}; N, M) = \frac{\partial}{\partial \mathcal{E}} L(\mathcal{E}; N, M) = a(N, M) \cdot \frac{M - 1}{2} \cdot N^{\frac{M-1}{2}} \cdot \mathcal{E}^{\frac{M-3}{2}} \qquad (29)$$

The corresponding pseudo probability density $P(\mathcal{E}; N, M)$ is given by (12). The normalization requirement for these functions becomes:

$$1 = \int_0^{\mathcal{E}_{max}} P(\mathcal{E}; N, M)d\mathcal{E} = \int_0^{\mathcal{E}_{max}} g(\mathcal{E}; N, M) \cdot exp(\mu(N, M) - \mathcal{E})d\mathcal{E} \qquad (30)$$

The $\mathcal{E}_{max}$ value is obtained from (9) by having microstate $j$ with lowest probability $p_{min} = \min\limits_{i \in G}\{p_i\}$ acquire maximum population: $n_{max} = N$; $n_{i \neq j} = 0$. From (9) as $N \to \infty$:

$$\mathcal{E}_{max}\big((n_i); N, (p_i)\big) \cong -N \cdot \ln p_{min} \tag{31}$$

For probabilities (5):

$$\mathcal{E}_{max}\big((n_i); N, M\big) \cong N \cdot \ln M \tag{32}$$

From (31) $\mathcal{E}_{max} \to \infty$ as $N \to \infty$. That allows replacing $\mathcal{E}_{max}$ in the upper limit of integral in (30) with $\infty$. I get (Forbes, Evans, Hastings, & Peacock, 2010) the expression for function $a(N, M)$ in (27) as:

$$a(N, M) = \left[ e^{\mu(N,M)} \cdot N^{\frac{M-1}{2}} \cdot \int_0^\infty \mathcal{E}^{\frac{M-1}{2}} e^{-\mathcal{E}} d\mathcal{E} \right]^{-1} = \frac{e^{-\mu(N,M)}}{N^{\frac{M-1}{2}} \Gamma\left(\frac{M+1}{2}\right)} \tag{33}$$

Using (33) and (17) allows rewriting (27) as

$$L(\mathcal{E}; N, M) = \frac{\mathcal{E}^{\frac{M-1}{2}}}{\Gamma\left(\frac{M+1}{2}\right)} e^{-\mu(N,M)} = \frac{(2\pi N \mathcal{E})^{\frac{M-1}{2}}}{M^{\frac{M}{2}} \cdot \Gamma\left(\frac{M+1}{2}\right)} = \frac{1}{\sqrt{2}} \left(\frac{2}{M}\right)^{\frac{M}{2}} \cdot V\big(\sqrt{N\mathcal{E}}; M - 1\big) \tag{34}$$

, where $\qquad V\big(\sqrt{N\mathcal{E}}; M - 1\big) = \dfrac{(\pi N \mathcal{E})^{\frac{M-1}{2}}}{\Gamma\left(\frac{M+1}{2}\right)} \qquad$ is the volume of $(M - 1)$ –dimensional sphere of radius $\sqrt{N\mathcal{E}}$.

The number $n(\mathcal{E})$ of distinct values of $\mathcal{E}$ in $N \to \infty$ limit can be estimated from (34) and (14) as

$$n(\mathcal{E}) = \frac{L(\mathcal{E}; N, M)}{M!} = \frac{(2\pi N \mathcal{E})^{\frac{M-1}{2}}}{M^{\frac{M}{2}} \cdot \Gamma\left(\frac{M+1}{2}\right) \Gamma(M + 1)} \tag{35}$$

From (35) one can approximately enumerate distinct energy levels $\mathcal{E}_n$ by "quantum number" $n$:

$$\mathcal{E}_n = \left[ \Gamma\left(\frac{M+1}{2}\right) \Gamma(M + 1) e^{\mu(N,M)} \cdot n \right]^{\frac{2}{M-1}} = \frac{M}{2\pi N} \left[ \Gamma\left(\frac{M+1}{2}\right) \Gamma(M + 1) M^{\frac{1}{2}} \cdot n \right]^{\frac{2}{M-1}} \tag{36}$$

From (29) the pseudo density $g(\mathcal{E}; N, M)$ of distinguishable states of statistical ensemble is

$$g(\mathcal{E}; N, M) = \frac{\partial}{\partial \mathcal{E}} L(\mathcal{E}; N, M) = \frac{\mathcal{E}^{\frac{M-3}{2}} e^{-\mu(N,M)}}{\Gamma\left(\frac{M-1}{2}\right)} \tag{37}$$

I use condition (13) to define effective $\mathcal{E}_{max}^{eff}$ value:

$$L(N, M) = L\big(\mathcal{E}_{max}^{eff}; N, M\big) = \frac{\mathcal{E}_{max}^{eff}{}^{\frac{M-1}{2}}}{\Gamma\left(\frac{M+1}{2}\right)} e^{-\mu(N,M)} = \frac{(N + M - 1)!}{N! \, (M - 1)!} \tag{38}$$

Figure 7 show $L(\mathcal{E}; N, \{p_i\})$ calculated from exact expressions (4), (9), and from formula (34). From (11), (17), and (37) the pseudo probability density function of statistical ensemble in thermodynamic limit is

$$P(\mathcal{E}; N, M) = \frac{\mathcal{E}^{\frac{M-3}{2}} e^{-\mathcal{E}}}{\Gamma\left(\frac{M-1}{2}\right)} = \gamma_{b,c}(\mathcal{E}); \qquad b = 1; \ c = \frac{M-1}{2} \tag{39}$$

, where $\gamma_{b,c}(\mathcal{E})$ is the probability density function of *gamma* (Forbes, Evans, Hastings, & Peacock, 2010) distribution with scale parameter $b = 1$, and shape parameter $c = (M - 1)/2$.
I calculate moments of $\mathcal{E}$:

Mean:
$$\overline{\mathcal{E}} = \sum_{(n_i)} \mathcal{E}\big((n_i); N, M\big) \cdot P\big((n_i); N, M\big) \tag{40}$$

Variance:
$$\sigma_{\mathcal{E}}^2 = \sum_{(n_i)} \big(\mathcal{E}\big((n_i); N, M\big) - \overline{\mathcal{E}}\big)^2 \cdot P\big((n_i); N, M\big) \tag{41}$$

$r^{th}$ moment
about mean:
$$\kappa_r(N, M) = \sum_{(n_i)} \big(\mathcal{E}\big((n_i); N, M\big) - \overline{\mathcal{E}}\big)^r \cdot P\big((n_i); N, M\big) \tag{42}$$

The sums in (40-42) are over all combinations of $(n_i)$ satisfying (1), i.e. over all partitions of $N$. Expression (39) allows explicit calculation of all moments of $\mathcal{E}$ in thermodynamic limit. From (39) the mean value $\overline{\mathcal{E}}$, the variance $\sigma_{\mathcal{E}}^2$, and the third moment $\kappa_3$ are:

$$\overline{\mathcal{E}} = \frac{M - 1}{2} \tag{43}$$

$$\sigma_{\mathcal{E}}^2 = \frac{M - 1}{2} \tag{44}$$

$$\kappa_3 = M - 1 \tag{45}$$

Figure 8 shows calculations of mean value $\overline{\mathcal{E}}$, the variance $\sigma_{\mathcal{E}}^2$, and the third moment $\kappa_3$ from the exact expressions (40-42) for the moments and (4) for the probability mass function. It demonstrates how these values asymptotically approach thermodynamic limit values (43-45) as $N \cdot p_i \to \infty$, i.e. as $t \to \infty$ where $t$ is the proper time (Viznyuk, 2011).

I shall demonstrate how the presented model correlates with some known constructs. Consider one-dimensional quantum harmonic oscillator. Its energy levels (Griffiths, 2005) are given by:

$$\mathcal{E}_n = \left(n + \frac{1}{2}\right) \cdot \hbar\omega \tag{46}$$

, where $\omega$ is the base frequency, and n $= 0,1,2 \ldots$ . Energy levels (46) are equally-spaced. In my model similar pattern is exhibited by energy levels of statistical ensemble of cardinality $M = 3$, as shown on Figure 1. From (18) in thermodynamic limit approximation, the energy of statistical ensemble can be written as:

$$\mathcal{E} = \sum_{i \in G} \frac{\Delta_i{}^2}{2\overline{n}} \tag{47}$$

, where
$$\Delta_i = n_i - \overline{n} \quad ; \quad \sum_i \Delta_i = 0 \quad ; \quad \overline{n} = \frac{N}{M} \tag{48}$$

From above, the energy levels of statistical ensemble of cardinality $M = 3$ are:

$$\mathcal{E}_k = \frac{L_k}{N} \tag{49}$$

, where $L_k$ are *Loeschian numbers* (Sloane, OEIS sequence A003136, 2015). If I designate the base frequency $\omega = 2/(N\hbar)$, I can write the comparison table of the first few energy levels of quantum harmonic oscillator and statistical ensemble of cardinality $M = 3$ in units of $\hbar\omega/2$:

| quantum harmonic oscillator | ■ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| statistical ensemble of cardinality $M = 3$ | 0 | 1 | 3 | 4 | 7 | 9 | 12 | 13 | 16 | ■ | 19 | 21 | ■ | 25 | 27 |

, where black boxes designate missing energy levels. In the second row the energy levels shown in shaded boxes are only realized for modes with $mod(N, 3) > 0$; and energy levels shown in white boxes are realized for modes with $mod(N, 3) = 0$. Here $mod(N, 3)$ is the remainder of division of $N$ by 3.

Consider another classic quantum mechanical example: particle of mass $m$ in a box of size $L$. Its energy levels (Griffiths, 2005) are given by:

$$\mathcal{E}_n = \frac{h^2}{8mL^2} n^2 \quad ; \quad n = 1, 2, 3 \dots \qquad (50)$$

In my model similar energy spectrum is exhibited by statistical ensemble of cardinality $M = 2$, as shown on Figure 2. From (47) the energy levels of statistical ensemble of cardinality $M = 2$ in thermodynamic limit approximation are:

$$\mathcal{E}_n = \frac{n^2}{2N} = \frac{h^2}{8mL^2} n^2 \quad ; \quad n = 0, 1, 2 \dots \qquad (51)$$

, where $\quad m = N \cdot \left(\frac{h}{2L}\right)^2 \quad$ is to be considered as an *effective mass* of the particle. $\qquad$ (52)

Energy levels (51) with even $n$ are only possible when $N$ is even, and energy levels with odd $n$ are only possible when $N$ is odd. With ½ probability the lowest energy level is $\mathcal{E} = \mathcal{E}_0 = 0$, and with ½ probability it is $\mathcal{E} = \mathcal{E}_1 = 1$, in units of $h^2/(8mL^2)$.

## 3. THERMODYNAMIC ENSEMBLE

The statistical ensemble considered in previous section represents a single copy of underlying system, with mode $\boldsymbol{k} = (n_{i \in \boldsymbol{G}})$ uniquely identifying the state. In this section I consider observation as a random pick of underlying system from a collection of systems, each represented by its own statistical ensemble of cardinality $M$. I call such collection of systems *thermodynamic ensemble*. I designate $\{\boldsymbol{k}\}$ the set of *modes* a system may occupy; $K$ the total number of systems, and $K_k$ the number of systems in mode $\boldsymbol{k}$:

$$\sum_{\{k\}} K_k = K \qquad (53)$$

I designate $exp(\rho(N))$ the probability for a system to be in any mode with total population of microstates $N$. Then, from (11), the probability for a system to be in mode $\boldsymbol{k}$ is:

$$p_k = exp(\rho(N) + \mu(N) - \mathcal{E}(\boldsymbol{k})) = exp(\rho_N + \mu_N - \mathcal{E}_{\boldsymbol{k}}) \qquad (54)$$

I consider systems in the same mode $\boldsymbol{k}$ indistinguishable to the observer. The probability mass function of distribution of modes among systems is

$$P\big((K_k); K, (p_k)\big) = K! \prod_{\{k\}} \frac{p_k^{K_k}}{K_k!} \qquad (55)$$

The objective is to find the most probable distribution $(K_k)$. For standalone systems the most probable distribution is the one which maximizes (55), i.e.

$$K_k = K \cdot p_k \qquad (56)$$

I consider systems to be part of some bigger system in a certain state. That imposes conditions on distribution of modes among systems, so relations (43-45), (56) may no longer hold. I consider one of the possible conditions and show how it leads to the notion of temperature. Let the state of the bigger system be such that the average energy of the systems in thermodynamic ensemble is $\overline{\mathcal{E}}$, which may be different from the average energy of standalone systems given by (43). Then:

$$\overline{\mathcal{E}} \cdot K = \sum_{\{k\}} K_k \cdot \mathcal{E}_{\boldsymbol{k}} \tag{57}$$

To find the most probable distribution of modes $(K_k)$ I shall maximize logarithm of (55) using method of Lagrange multipliers (Vapnyarskii, 2001; Huang, 2001) with conditions (53) and (57):

$$\ln P\big((K_k); K, (p_k)\big) = \ln \Gamma(K + 1) + \sum_{\{k\}} [K_k \cdot \ln p_k - \ln \Gamma(K_k + 1)]$$

$$= \ln \Gamma(K + 1) + \sum_{\{k\}} [K_k \cdot (\rho_N + \mu_N - \mathcal{E}_{\boldsymbol{k}}) - \ln \Gamma(K_k + 1)] \tag{58}$$

From (58), (57), (53) I obtain the following equation involving Lagrange multipliers $\alpha$ and $\beta$:

$$\Psi_0(K_k + 1) = \rho_N + \mu_N - (1 + \alpha) \cdot \mathcal{E}_{\boldsymbol{k}} - \beta \tag{59}$$

, where $\Psi_0$ is *digamma* function, and $\alpha$ and $\beta$ are to be determined by solving (59) for $K_k$:

$$K_k = \Psi_0^{-1}\left(\rho_N + \mu_N - \frac{\mathcal{E}_{\boldsymbol{k}}}{T} - \beta\right) - 1 \tag{60}$$

, and by plugging $K_k$ from (60) into (57) and (53). In (60) $\Psi_0^{-1}$ is the inverse digamma function, and $1/T = 1 + \alpha$. The parameter $T$ is commonly known as *temperature*.

Since the number of systems $K_k$ in mode $\boldsymbol{k}$ cannot be negative, expression (60) effectively limits modes which can be present in most probable distribution to those satisfying

$$\rho_N + \mu_N - \frac{\mathcal{E}_{\boldsymbol{k}}}{T} - \beta + \gamma \geq 0 \tag{61}$$

, where $\gamma \cong 0.577215665$ is *Euler–Mascheroni* constant. Using approximation (Abramowitz, 1972): $exp\big(\Psi_0(K_k + 1)\big) \cong K_k + 1/2$, I rewrite (60) as:

$$K_k \cong exp\left(\rho_N + \mu_N - \frac{\mathcal{E}_{\boldsymbol{k}}}{T} - \beta\right) - \frac{1}{2} \tag{62}$$

Presence of $-\frac{1}{2}$ term in (62) leads to a computationally horrendous task of calculating $\beta$ and $T$, because the summation in (53) and (57) has to be only performed for modes satisfying (61). I shall leave the exact computation to a separate exercise, and make a shortcut, by ignoring $-\frac{1}{2}$ term in (62). This approximation is equivalent to a common postulate[2] (Landau & Lifshitz, 1980) that the number of systems in mode $\boldsymbol{k}$ is proportional to $exp(-\mathcal{E}_k/T)$. The shortcut allows calculation of Lagrange multiplier $\beta$ from (53):

$$exp(-\beta) = \frac{K}{Z(T)} \qquad \text{, where} \qquad Z(T) = \sum_{\{k\}} exp\left(\rho_N + \mu_N - \frac{\mathcal{E}_{\boldsymbol{k}}}{T}\right) \tag{63}$$

Using expression (37), the partition function $Z(T)$ in (63) can be evaluated as:

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[2] While widely used this postulate has rather unphysical consequence that there is a non-zero probability of finding a system in a mode with arbitrary large energy. Another consequence is the divergence of partition function for some constructs, e.g. hydrogen electronic levels (Strickler, 1966).

$$Z(T) = \sum_{N=1}^{\infty} exp(\rho_N) \sum_{\{k\}_N} exp\left(\mu_N - \frac{\mathcal{E}_k}{T}\right) = \sum_{N=1}^{\infty} exp(\rho_N) \int_0^{\infty} g(\mathcal{E}; N, M) \, exp\left(\mu_N - \frac{\mathcal{E}}{T}\right) d\mathcal{E}$$

$$= \sum_{N=1}^{\infty} exp(\rho_N) \int_0^{\infty} \frac{\mathcal{E}^{\frac{M-3}{2}}}{\Gamma\left(\frac{M-1}{2}\right)} exp\left(-\frac{\mathcal{E}}{T}\right) d\mathcal{E} = T^{\frac{M-1}{2}} \sum_{N=1}^{\infty} exp(\rho_N) = T^{\frac{M-1}{2}} \tag{64}$$

The equation (57) then becomes
$$\overline{\mathcal{E}} = T^2 \cdot \frac{\partial}{T} \ln Z = \frac{M-1}{2} \cdot T \tag{65}$$

Eq. (65) is the familiar relation (Huang, 2001) between average per-particle energy and temperature in $(M-1)$-dimensional ideal Maxwell-Boltzmann gas. Thermodynamic entropy $S_T$ can be evaluated as:

$$S_T = -\sum_{\{k\}} P_k(T) \cdot \ln P_k(T) = \frac{M-1}{2} \ln(eT) - \sum_{\{k\}} (\rho_N + \mu_N) \cdot P_k(T)$$

$$= \frac{M-1}{2} \ln(eT) - \sum_{N=1}^{\infty} (\rho_N + \mu_N) exp(\rho_N) \sum_{\{k\}_N} \frac{exp\left(\mu_N - \frac{\mathcal{E}_k}{T}\right)}{Z(T)} \tag{66}$$

$$= \frac{M-1}{2} \ln(eT) - \sum_{N=1}^{\infty} (\rho_N + \mu_N) \cdot exp(\rho_N)$$

, where
$$P_k(T) = \frac{K_k}{K} = \frac{exp\left(\rho_N + \mu_N - \frac{\mathcal{E}_k}{T}\right)}{Z(T)} \tag{67}$$

With expression (17) for $\mu_N$, in thermodynamic limit, I rewrite (66) as

$$S_T = \frac{M-1}{2} \ln(2\pi eT) - \frac{M}{2} \ln M + S_N + \frac{M-1}{2} \sum_N exp(\rho_N) \ln N \tag{68}$$

, where
$$S_N = -\sum_N \rho_N \cdot exp(\rho_N) \tag{69}$$

To calculate $S_N$ I have to make an assumption on $exp(\rho_N)$ distribution. As a possible example, I shall assume the number $N = \sum_{i \in G} n_i$ of microstates for a system in thermodynamic ensemble is Poisson-distributed around mean $\overline{N} \gg 1$ value. Therefore, for $S_N$ I can use expression for the entropy of Poisson distribution (Evans, Boersma, Blachman, & Jagers, 1988):

$$S_N \cong \frac{1}{2} \ln 2\pi e \overline{N} \tag{70}$$

I also use the following:
$$\sum_N exp(\rho_N) \ln N = \overline{\ln N} \cong \ln \overline{N} \tag{71}$$

With (70), (71) I finally obtain:
$$S_T = \frac{M}{2} \ln\left(2\pi e \frac{\overline{N}}{M}\right) + \frac{M-1}{2} \ln T \tag{72}$$

In case of $M = 4$, i.e. for $(M-1) = 3$ degrees of freedom, the expression (72) turns into equivalent of Sackur-Tetrode equation (Huang, 1987) for entropy of ideal gas. For thermodynamic entropy of a standalone system, instead of (66-72) from (17) and (43) I have:

$$S = -\sum_{\{k\}} P_k \cdot \ln P_k = -\sum_{\{k\}} exp(\mu_N - \mathcal{E}_k) \cdot (\mu_N - \mathcal{E}_k) = \bar{\mathcal{E}} - \mu_N$$

$$= \frac{M-1}{2} - \frac{M}{2}\ln M + \frac{M-1}{2}\ln 2\pi N = \frac{M}{2}\ln\left(2\pi e\frac{N}{M}\right) - \frac{1}{2}\ln 2\pi eN \qquad (73)$$

Thermodynamic entropy (72) per system in thermodynamic ensemble is larger than entropy (73) of a standalone system by term (70) plus the temperature-related term. The increase in entropy by $S_N$ happens because of the spread in values of $N$, i.e. in *age* of the systems. The increase in entropy by temperature-related term $\frac{M-1}{2}\ln T$ is due to the spread in energies of the systems. The non-zero thermodynamic entropy of a standalone system implies its state is unknown prior to observation, for each observation. If the state is known prior to observation, e.g., as a result of prior observation, then thermodynamic entropy is zero. It still leaves non-zero *microstate entropy* (Viznyuk, 2011). Using (1) I rewrite (73) in terms of proper time $t$ as:

$$S(t; M) = S_0(M) + \frac{M-1}{2}t \qquad , \text{ where } \qquad S_0(M) = \frac{M-1}{2}\ln 2\pi e - \frac{M}{2}\ln M \qquad (74)$$

The expression for $S_0(M)$ in (74) was derived in thermodynamic limit, i.e. when $N \to \infty$. When $N = 1$ (i.e. when $t = 0$) $S = \ln M$. By comparing $S_0(M)$ to $\ln M$ (Figure 9) I see that $S_0(M)$ fairly closely matches $\ln M$ except when $M$ is large enough, in which case thermodynamic limit approximation for the given $N$ becomes less valid anyhow. Therefore, I can replace $S_0(M)$ with $\ln M$ in (74) and obtain thermodynamic entropy of a standalone system as:

$$S(t; M) = \ln M + \frac{M-1}{2}t \qquad (75)$$

Eq. (75) establishes [linear] relation between thermodynamic entropy and proper time, as yet another manifestation of the Second Law of Thermodynamics (SLT). Previously, SLT has been demonstrated in the context of time model (Viznyuk, 2011) using numeric calculation of *microstate entropy*.

The expression for $Z(T)$ in (64) has been derived in thermodynamic limit approximation, i.e. when $N \to \infty$. It means there must be large number of energy levels included in sum (63), i.e. temperature $T$ cannot be too small. Therefore, the expressions (64-65) are only valid for $T \gg \Delta\mathcal{E}$, where $\Delta\mathcal{E}$ is the characteristic spacing between energy levels.

For statistical ensemble of cardinality $M = 3$ the approximately evenly-spaced energy levels (see Figure 1) allow for more accurate expression for partition function. For a mode with given $N$ the characteristic spacing between energy levels in the limit $N \to \infty$ is:

$$\frac{M!}{g(\mathcal{E}; N, M)} = 6 \cdot exp(\mu_N) = \frac{18\sqrt{3}}{2\pi N} \cong \frac{5}{N} \qquad (76)$$

With time (1) increment, the system transitions from a mode with $N$ to a mode with $N + 1$, and between adjacent energy levels of the mode with $N$ and the mode with $N + 1$. In previous section I have shown the characteristic spacing between adjacent energy levels of combined modes for statistical ensemble of cardinality $M = 3$ is $\hbar\omega = 2/N$. If $mod(N, 3) = 0$ the first 17 energy levels in units of $\hbar\omega/2 = 1/N$ and their degeneracy are:

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{E}_k$ | 0 | 3 | 9 | 12 | 21 | 21 | 27 | 36 | 39 | 39 | 48 | 57 | 57 | 63 | 63 | 75 | 81 |
| $g_k$ | 1 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |

When $mod(N, 3) > 0$ the first 16 energy levels in the same units and their degeneracy are:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{E}_k$ | 1 | 4 | 7 | 13 | 16 | 19 | 25 | 28 | 31 | 37 | 43 | 49 | 52 | 61 | 64 | 67 |
| $g_k$ | 3 | 3 | 6 | 6 | 3 | 6 | 3 | 6 | 6 | 6 | 6 | 9 | 6 | 6 | 3 | 6 |

If I were to use (46) as an approximation for the combined energy levels above, the average energy of modes with given $N$ would be equal (Planck's law):

$$\overline{\mathcal{E}}_N = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{T}\right) - 1} \tag{77}$$

From here $\overline{\mathcal{E}}_N = T \cdot \left[1 + \frac{(\hbar\omega)^2}{12T^2} + O\left(\left(\frac{\hbar\omega}{T}\right)^3\right)\right]$ when $\hbar\omega \ll T$. The relation $\overline{\mathcal{E}}_N / \hbar\omega$ is sometimes referred to as the average number of photons in a mode (Kittel & Kroemer, 1980). In my model the notion of a photon is meaningless. The quantized energy levels (9,47) make transitions between modes appear as absorption or emission of particles.

There is a continuing controversy (Kish, 2015) about the first term in (77). $\hbar\omega/2$ term results in infinite *zero-point* energy of the field if all modes $0 \le \hbar\omega \le \infty$ are summed up. The common "solution" to this problem is to ignore $\hbar\omega/2$ term when calculating radiation density, via technique called renormalization. The problem does not exist in my model since $\hbar\omega$ has an upper limit (76).

The notion of a temperature, like the notions of energy, of force, and many other in modern physics lacks clear definition. For example, Wikipedia (Temperature, 2016) provides an example of a definition void of any meaning: *Temperature is the transfer of thermal energy between objects*. Other (Cool Cosmos, 2016) seem to do somewhat better: *Temperature is a measure of the average heat or thermal energy of the particles in a substance*.

In this section I have arrived at the notion of temperature as the ratio of average energy (65) of the systems in thermodynamic ensemble to the expectation value (43) of energy of a randomly chosen standalone system. Thus, the temperature is a measure of *ergodicity* with respect to energy as statistical parameter. If $T = 1$ then thermodynamic ensemble is ergodic.

## 4. DYNAMICS OF OBSERVABLES

In this section I discuss dynamics of observables in the context of time model (Viznyuk, 2011). An observable value can be any function of canonical variables (22) or any function of population numbers $(n_i)$. Such function represents transformation by the measuring device from variables which pertain to underlying system only, to observable values. Within the scope of this section I consider characteristic timescales of transformation by the measuring device much shorter than the timescale of exponential decline (24). This lends the traditional interpretation of conservation of energy law. Special interest present observable values $\{z_i\}$ obtained as an orthogonal projection $\Theta$ of canonical variables $\{x_i\}$ in (22):

$$z = \Theta^T \cdot x \tag{78}$$

Transformation (78) preserves quadratic form (23) which means the observable values $\{z_i\}$ have familiar from classical mechanic relation to energy. The orthogonal transformation $\Theta$ in (78) represents the measuring device. The state of measuring device may reflect the state of underlying system. Therefore, the orthogonal transformation $\Theta$ may depend on $x$ and, possibly, proper time: $\Theta = \Theta(t, x)$. Any orthogonal matrix $\Theta$ can be expressed (Youla, 1961) as a matrix exponent of a real skew-symmetric matrix $A$:

$$\Theta = exp(A) \quad \text{, where } A_{ij} = -A_{ji} \tag{79}$$

To demonstrate the possible behavior of observed values, I used the simplest form of $\boldsymbol{A}$ as

$$A_{ij} = W \cdot (x_i - x_j) ; \qquad i, j > 1 \tag{80}$$

, where $W$ is some non-zero real number[3], and:

$$A_{1j} \equiv 0 \quad ; \quad A_{i1} \equiv 0 \quad \forall\, i, j \tag{81}$$

Condition (81) ensures transformation (78) is restricted to hyper-plane (1). Without condition (81), due to $x_1 \equiv 0$ in (22), transformation (78) would result in linearly dependent components of vector $\boldsymbol{z}$, while I'm interested in irreducible representation. Therefore, I consider all transformations are restricted to hyper-plane (1) and all matrices are of rank $M - 1$, dropping $x_1 \equiv 0$ component.

I used transformation (78) with (79-81) and time model defined in (Viznyuk, 2011) to numerically calculate dynamics of observable values $\{z_i\}$ and of energy $\mathcal{E}$. The MATLAB code used for calculations and graphs is http://phystech.com/download/ensemble_dynamics12.m. Figure (10) demonstrates time dynamics of $\{z_i\}$ for statistical ensemble of cardinality $M = 3$. The observable values $z_2$ and $z_3$ exhibit behavior similar to a damped harmonic oscillator, where $z_2$ represents the coordinate, and $z_3$ the momentum. The oscillations of $z_2$ and $z_3$ are shifted by $\pi/2$ with respect to each other, as are the coordinate and momentum of a harmonic oscillator. The corresponding time dynamics of energy is shown on Figure 12.

The choice of skew-symmetric matrix (80) is rather arbitrary. I only used it to demonstrate possible time dynamics of observables within the context of the model.

I want to explicitly show the temporal dependence of observables, and then compare it to numeric simulation on Figure 10. To do that I note that arbitrary real skew-symmetric matrix $\boldsymbol{A}$ can be reduced (Youla, 1961; Voronov, 2003) to a block-diagonal form $\mathbf{D}$ by an orthogonal transformation $\boldsymbol{O}$:

$$\boldsymbol{A} = \boldsymbol{O} \cdot \boldsymbol{D} \cdot \boldsymbol{O}^T \tag{82}$$

, where

$$\boldsymbol{D} = \begin{pmatrix} 0 & 0 & 0 & & & \\ 0 & 0 & \alpha_1 & \cdots & & 0 \\ 0 & -\alpha_1 & 0 & & & \\ & \vdots & & \ddots & & \vdots \\ & & & & 0 & \alpha_n \\ & 0 & & \cdots & -\alpha_n & 0 \end{pmatrix}$$

, and $\alpha_i = \alpha_i(t, \boldsymbol{x})$ are real $\tag{83}$

Plugging (82) into (79) I obtain from (78):

$$\boldsymbol{z} = \boldsymbol{O} \cdot exp(\boldsymbol{D}^T) \cdot \boldsymbol{O}^T \cdot \boldsymbol{x} \tag{84}$$

, where

$$exp(\boldsymbol{D}^T) = \begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & cos(\alpha_1) & -sin(\alpha_1) & \cdots & & 0 \\ 0 & sin(\alpha_1) & cos(\alpha_1) & & & \\ & \vdots & & \ddots & & \vdots \\ & & & & cos(\alpha_n) & -sin(\alpha_n) \\ & 0 & & \cdots & sin(\alpha_n) & cos(\alpha_n) \end{pmatrix} \tag{85}$$

I re-write (84) as

$$\boldsymbol{y} = \boldsymbol{O}^T \cdot \boldsymbol{z} = exp(\boldsymbol{D}^T) \cdot \boldsymbol{O}^T \cdot \boldsymbol{x} = exp(\boldsymbol{D}^T) \cdot \boldsymbol{u} \tag{86}$$

, where I introduced new knowledge vector $\boldsymbol{y} = \boldsymbol{O}^T \cdot \boldsymbol{z}$ and new state vector $\boldsymbol{u} = \boldsymbol{O}^T \cdot \boldsymbol{x}$. Action of $exp(\boldsymbol{D}^T)$ operator on vector $\boldsymbol{u}$ in (86) constitutes transformation by the measuring device. Vector

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[3] $W$ has to be non-zero real number to ensure $\boldsymbol{\Theta}$ is an orthogonal transformation

$\boldsymbol{u}$ represents the underlying system in *eigenbasis* of the measuring device. In this basis the action of measuring device is reduced to rotations in *2-D* orthogonal eigenspaces, as evident from (85).

As I mentioned earlier, the rank of matrices $\boldsymbol{D}, \boldsymbol{O}$ and of vectors $\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}, \boldsymbol{u}$ is $M - 1$. Therefore, transformation (86) takes especially simple form in case of $M = 3$:

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \boldsymbol{u} \ , \text{where } \ \alpha = \alpha(t, \boldsymbol{u}) \ , \text{and } \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{87}$$

When $M = \{1, 2\}$ the transformation is trivial: if $M = 1$ then $\boldsymbol{y} \equiv 0$; if $M = 2$ then $\boldsymbol{y} = 1 \cdot \boldsymbol{u}$. Transformation (87) can be expressed in complex notation using real components $u_1, u_2, y_1, y_2$:

$$\boldsymbol{y} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \cdot \boldsymbol{u} \ \ , \text{where } \ \boldsymbol{u} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} u_1 + iu_2 \\ u_1 - iu_2 \end{pmatrix} \ , \text{and } \ \boldsymbol{y} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} y_1 + iy_2 \\ y_1 - iy_2 \end{pmatrix} \tag{88}$$

In case of $M > 3$ one can convert to complex notation by combining pairs of real $\{u_i\}, \{y_i\}$ components corresponding to *2-D* eigenspaces in (85) into complex numbers as in (88). In case of even $M$ there will be one real-only component $u_i$ left un-transformed. In complex notation (85) can be written as a unitary matrix $\boldsymbol{U}$;

for even $M$:
$$\boldsymbol{U} = \begin{pmatrix} 1 & & & & \\ & e^{-i\alpha_1} & & \cdots & 0 \\ & & e^{i\alpha_1} & & \\ & \vdots & & \ddots & \vdots \\ & & & & e^{-i\alpha_n} \\ & 0 & & \cdots & e^{i\alpha_n} \end{pmatrix} \ , \text{where } \ \alpha_i = \alpha_i(t, \boldsymbol{u}) \tag{89}$$

, and for odd $M$:
$$\boldsymbol{U} = \begin{pmatrix} e^{-i\alpha_1} & & \cdots & 0 \\ & e^{i\alpha_1} & & \\ \vdots & & \ddots & \vdots \\ & & & e^{-i\alpha_n} \\ 0 & & \cdots & e^{i\alpha_n} \end{pmatrix} \tag{90}$$

Transformation (86) can now be expressed in complex notation as:

$$\boldsymbol{y} = \boldsymbol{U}^\dagger \cdot \boldsymbol{u} \tag{91}$$

In complex notation orthogonal matrix (85) takes diagonal form and becomes unitary matrix (89-90). Operating with unitary matrices in diagonal form is easier than with orthogonal matrix (85), which is the reason to use complex notation. It is tempting to eliminate the seeming redundancy in pairs of complex eigenvalues in (89-90), and in their corresponding eigenvectors (88) and reduce the rank of the vectors and matrices by half. I hesitate to do that because it will introduce ambiguity about the rank of underlying system. For example, the following unitary transformation and associated knowledge vector would then apply to underlying system of cardinality $M = 4$ as well as $M = 5$ (if rotation angle for the first eigenspace is 0):

$$\boldsymbol{y} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \cdot \boldsymbol{u}$$

Expressions (87-88) represent orthogonal transformation of observation basis for statistical ensemble of cardinality $M = 3$. In previous section I have shown that $M = 3$ case corresponds to a harmonic oscillator, i.e. *free field*. Expressions (84-86) or equivalent expressions (89-91) define *measurement* (78) as *2-D* rotations (87-88) of observation basis within *2-D* eigenspaces of the measuring device. This is the fundamental principle behind QM [in Heisenberg interpretation] describing evolution of a system as rotation (i.e. unitary transformation) of observation basis.

Consider scenario where eigenspaces of the measuring device do not explicitly depend on time, i.e. $\partial \boldsymbol{O}(t, \boldsymbol{x})/\partial t = 0$ in (86). Then, $\partial \boldsymbol{u}/\partial t = -\boldsymbol{u}$, and from (89-91) I derive the following differential equation for vector $\boldsymbol{y}$:

$$\frac{\partial \boldsymbol{y}}{\partial t} = \left[\frac{\partial}{\partial t} \boldsymbol{U}^\dagger\right] \cdot \boldsymbol{u} + \boldsymbol{U}^\dagger \cdot \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{\Phi}^\dagger \cdot \boldsymbol{y} \tag{92}$$

, where

$$\boldsymbol{\Phi} = \begin{pmatrix} 0 & & & & \\ & -i\frac{\partial \alpha_1}{\partial t} - 1 & & \cdots & & 0 \\ & & i\frac{\partial \alpha_1}{\partial t} - 1 & & & \\ & \vdots & & \ddots & & \vdots \\ & & & & -i\frac{\partial \alpha_n}{\partial t} - 1 & \\ & 0 & & \cdots & & \\ & & & & & i\frac{\partial \alpha_n}{\partial t} - 1 \end{pmatrix} \tag{93}$$

$$= \begin{pmatrix} 0 & & & & \\ & -i\omega_1 - 1 & & \cdots & & 0 \\ & & i\omega_1 - 1 & & & \\ & \vdots & & \ddots & & \vdots \\ & & & & -i\omega_n - 1 & \\ & 0 & & \cdots & & \\ & & & & & i\omega_n - 1 \end{pmatrix} \quad \text{for even } M$$

$$\boldsymbol{\Phi} = \begin{pmatrix} -i\frac{\partial \alpha_1}{\partial t} - 1 & & \cdots & & 0 \\ & i\frac{\partial \alpha_1}{\partial t} - 1 & & & \\ \vdots & & \ddots & & \vdots \\ & & & -i\frac{\partial \alpha_n}{\partial t} - 1 & \\ 0 & & \cdots & & \\ & & & & i\frac{\partial \alpha_n}{\partial t} - 1 \end{pmatrix} \tag{94}$$

$$= \begin{pmatrix} -i\omega_1 - 1 & & \cdots & & 0 \\ & i\omega_1 - 1 & & & \\ \vdots & & \ddots & & \vdots \\ & & & -i\omega_n - 1 & \\ 0 & & \cdots & & \\ & & & & i\omega_n - 1 \end{pmatrix} \quad \text{for odd } M$$

, where

$$\omega_i = \frac{\partial \alpha_i(t, \boldsymbol{u})}{\partial t} \tag{95}$$

Equation (92) is similar to Schrödinger's equation where Hamiltonian $\boldsymbol{\mathcal{H}} = i\boldsymbol{\Phi}^\dagger$ and Planck's constant $\hbar = 1$, and where $\omega_i$ are characteristic frequencies. However:

1. Equation (92) is for the *knowledge vector*, while Schrödinger equation is for the state vector (wave function). Wave function purports to represent the underlying system only. The evolution of wave function according to Schrödinger's equation is independent of representation, i.e. independent of the way the system is observed. For Schrödinger's equation different representations are unitarily equivalent. That contradicts to (Haag, 1955) theorem which states no such equivalence exists. In my model the evolution of knowledge

vector (91) is determined by propagator matrix $U$ which takes diagonal form (89-90) in representation of eigenvectors of the measuring device. Thus the evolution of knowledge vector depends on the measuring device. It is easy to prove no unitary mapping can be established between knowledge vector spaces for measuring devices with different eigenspaces. Consider two knowledge vectors $y_1$ and $y_2$ observed via two different devices in representation of eigenvectors of corresponding device. Then both propagator matrices $U_1$ and $U_2$ for $y_1$ and $y_2$ are diagonal. Suppose there is a unitary mapping $V$ such that $u_2 = V^\dagger u_1$ and $y_2 = V^\dagger y_1$. Then, from (91) the propagator matrices $U_1$ and $U_2$ for $y_1$ and $y_2$ must relate to each other as $U_2 = V^\dagger U_1 V$. If $U_1$ and $U_2$ are both diagonal, that is only possible if $U_1 = U_2$.

2. $\Phi$ matrix contains $-1$ components in diagonal elements which results in exponential decay of vector $y$, the same as exponential decay of canonical variables $x$ in (24), while Schrödinger equation describes superposition of non-decaying harmonic waves (Ward & Volkmer, 2006), which is evident if Schrödinger equation is written in energy eigenvector representation.

3. Characteristic frequencies $\omega_i$ are expected to exhibit exponential decay (24) as a result of exponential decay of canonical variables $x$.

Features 2,3 stated above are evident in numeric simulation shown on Figure 10, where knowledge vector and oscillation frequency decay with time as $t \to \infty$. Hence, $\omega_i(u = 0) = 0$ should be the minimum of functions $\omega_i(u)$. In the vicinity of $u = 0$, $\omega_i(u)$ is approximated by a quadratic form on $u$. The obvious choice of quadratic form is energy (23). Matrix $\Phi$ (93-95) must be invariant with respect to unitary transformations within eigenspaces of the measuring device, but not necessarily invariant with respect to unitary transformation of the whole knowledge vector space, per Haag's theorem. Therefore, each $\omega_i(u)$ can only depend on corresponding eigenspace component of energy:

$$\omega_i = \frac{\partial \alpha_i}{\partial t} = \frac{\mathcal{E}_i}{\hbar} \quad , \text{where} \quad \mathcal{E}_i = \frac{N}{2}\langle u_i | u_i \rangle = \frac{N}{2}\langle y_i | y_i \rangle \quad , \text{and} \quad \sum_i \mathcal{E}_i = \mathcal{E} \tag{96}$$

, and $\hbar$ is a constant of proportionality which I'm tempted to call Planck's constant.

Unlike (91), the equation (92) with (96) only contains reference to the knowledge vector $y$ and no reference to the state vector $u$ of underlying system.

The equations (91-92) for the knowledge vector bear familiar hallmarks of quantum physics. Consider e.g. wave-particle duality. The particle properties result from discreteness of probability mass function (4) and of energy spectrum (Table 1). Observable values change in "quantum" leaps, with the leap size decreasing as $\sim N^{-1}$ if $N \to \infty$, i.e. as underlying system approaches thermodynamic limit. The choice of observation basis can make quantum leaps between the states of underlying system look like emission and absorption of particles. In a case of underlying system represented by statistical ensemble of cardinality $M = 3$ the number of "particles" in the mode is linearly proportional to the energy of the system (Figure 1). For underlying system of cardinality $M > 3$ the energy is given by the sum (96) of energies of eigenspace fields, however, it is not linearly proportional to the number of particles, as can be seen from Figure 2.

Wave properties are exhibited through interference of knowledge vectors. Suppose the same underlying system, represented by vector $u$ in (91), is observed via two different devices, e.g. via two slits in double-slit experiment. First device is represented by propagator matrix $A$, and second device by $B$. The resultant knowledge vector $y$ is the sum of vectors $a$ and $b$:

$$y = a + b = (A^\dagger + B^\dagger) \cdot u \tag{97}$$

The field intensity, represented by $\mathcal{E}(\boldsymbol{y})$ is then:

$$\mathcal{E}(\boldsymbol{y}) = \frac{N \cdot \langle \boldsymbol{y}|\boldsymbol{y}\rangle}{2} = \frac{N \cdot \langle \boldsymbol{u}(A+B)|(A^\dagger + B^\dagger)\boldsymbol{u}\rangle}{2} = 2 \cdot \mathcal{E}(\boldsymbol{u}) + N \cdot \text{Re}\langle \boldsymbol{u}|AB^\dagger|\boldsymbol{u}\rangle \tag{98}$$

In order to obtain the desired result, I demand $A$, $B$ matrices have the same eigenspaces. In layman's terms, the interference occurs between wave components polarized along the same axis. Matrices $A$, $B$ diagonal in eigenspace representation are:

$$A = \begin{pmatrix} e^{-i\alpha_1} & & \cdots & & 0 \\ & e^{i\alpha_1} & & & \\ \vdots & & \ddots & & \vdots \\ & & & e^{-i\alpha_n} & \\ 0 & & \cdots & & e^{i\alpha_n} \end{pmatrix} \quad B = \begin{pmatrix} e^{-i\beta_1} & & \cdots & & 0 \\ & e^{i\beta_1} & & & \\ \vdots & & \ddots & & \vdots \\ & & & e^{-i\beta_n} & \\ 0 & & \cdots & & e^{i\beta_n} \end{pmatrix} \tag{99}$$

, where $\alpha_n$, $\beta_n$ are all real. Then (98) can be rewritten as below, in case of $M = 3$:

$$\mathcal{E}(\boldsymbol{y}) = 2 \cdot \mathcal{E}(\boldsymbol{u}) \cdot [1 + \cos(\alpha_1 - \beta_1)] \tag{100}$$

Expr. (100) is a characteristic intensity distribution in interference pattern in a double-slit experiment.

Consider another QM hallmark: violation of (Bell, 1964) inequalities. Within my framework the violation of Bell's inequalities can be understood without invoking the concept of wave function collapse. In a typical experiment (Aspect A., 1981; Hensen B., 2015) two entangled particles represent the same underlying system $\boldsymbol{u}$, which is being observed via two spatially separated devices $A$ and $B$. An observer *Alice* is attached to device $A$, and observer *Bob* is attached to device $B$. If *Alice* and *Bob* did not communicate via conventional channel, neither of them would know the result of other observer. The *statistical correlation* can only be detected when results (i.e. vectors $\boldsymbol{a}$ and $\boldsymbol{b}$) from each device converge to a single observer to form the resultant observation (97). The expression for the combined signal intensity is given by (98) and, in a particular case of $M = 3$, by (100). The target of experiments on violation of Bell's inequalities is the second term in (98,100). The $\cos()$ function in the second term manifests the violation (Bell, 1964). It shows double-slit experiment is about as good experiment on violation of Bell's inequalities as any other. Confusion of statistical correlation with causality in this context led some minds to bewilderment about *spooky action at a distance* (Einstein A., 1935).

If particles are not entangled, they form two independent underlying systems $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Then instead of (97-98) I have:

$$\boldsymbol{y} = \boldsymbol{a} + \boldsymbol{b} = A^\dagger \boldsymbol{u}_1 + B^\dagger \boldsymbol{u}_2 \tag{101}$$

$$\mathcal{E}(\boldsymbol{y}) = \mathcal{E}(\boldsymbol{u}_1) + \mathcal{E}(\boldsymbol{u}_2) + N \cdot \text{Re}\langle \boldsymbol{u}_1|AB^\dagger|\boldsymbol{u}_2\rangle \tag{102}$$

In case of $M = 3$ (102) becomes:

$$\mathcal{E}(\boldsymbol{y}) = \mathcal{E}(\boldsymbol{u}_1) + \mathcal{E}(\boldsymbol{u}_2) + N \cdot \langle \boldsymbol{u}_1|\boldsymbol{u}_2\rangle \cdot \cos(\alpha_1 - \beta_1) \tag{103}$$

When averaged over thermodynamic ensembles of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ the correlation term in (103) disappears. Expressions (97-103) imply $A$ and $B$ have the same eigenspaces, i.e. both devices measure the same observables.

If measurement $\boldsymbol{Q}(t)$ has been performed at time $t = 0$, and the result is $q_0$, what is the expectation value at time $t > 0$? The answer from QM is:

$$q(t) = \langle \boldsymbol{u}_0|E(t)\boldsymbol{Q}(0)E^\dagger(t)|\boldsymbol{u}_0\rangle \text{ , where } E(t) = exp\left(i\frac{H}{\hbar}t\right); \; H \text{ is Hamiltonian,} \tag{104}$$

, and $\boldsymbol{u}_0$ is the state of the system at $t = 0$. Is the system considered closed or open? In QM (104) implies the system is closed, as only a closed system can be described by the state vector $\boldsymbol{u}_0$. On

the other hand, if the system is closed, it has to be in an energy eigenstate. If $\boldsymbol{u}_0$ is also an energy eigenstate, then according to (104) $q(t) \equiv q_0 \; \forall \; t \geq 0$, i.e. a closed system is static. Current QM theory handles this paradox by considering system quasi-closed, i.e. described by a state vector, and $\boldsymbol{H}$-matrix with off-diagonal terms. Then $\boldsymbol{H}$ is not a true Hamiltonian of the system but so-called *interaction Hamiltonian*, and $\boldsymbol{u}_0$ is not an eigenstate of $\boldsymbol{H}$. There is no such paradox in my framework. But before I explain, let's define what open and closed system is:

- The system is *closed* in the given basis if all canonical state vectors of the system can be expressed via basis vectors. Otherwise, the system is *open*.

In my framework the statistical ensemble $(n_{i \in G})$ representing the underlying system can have the complete set of canonical variables (22) and observables (78) (i.e. basis) defined, in which the system is *closed*. However, with respect to other set of observables the same underlying system could be *open*. The concept of temperature was derived in previous section with this understanding. There the system is *open* with respect to a limited set of macroscopic observables such as *temperature*, since macroscopic observables do not define state vectors. Regardless of if the system is open or closed, it is not static in presented framework.

In QM (104) shows measurement $\boldsymbol{Q}(t) = \boldsymbol{E}(t)\boldsymbol{Q}(0)\boldsymbol{E}^\dagger(t)$ is obtained from measurement $\boldsymbol{Q}(t = 0)$ via unitary transformation $\boldsymbol{E}(t)$ of observation basis, in accordance with *Heisenberg picture* of QM formalism. Since the result of the measurement at $t = 0$ is one of the eigenvalues $q_s$ of operator $\boldsymbol{Q}(0)$, the state $\boldsymbol{u}_0$ of the system at $t = 0$ has to be one of the eigenstates of $\boldsymbol{Q}(0)$. Attempts to understand this fact have needlessly led *Copenhagen School* to the concept of *wave function collapse*. Below I shall derive the time dependence of $q(t)$ in (104), and then compare it with my framework. I designate $\boldsymbol{u}_s$ the eigenvectors of $\boldsymbol{Q}(0)$, corresponding to eigenvalues $q_s$. Then:

$$\boldsymbol{Q}(0) = \sum_s |\boldsymbol{u}_s\rangle q_s \langle \boldsymbol{u}_s| = \sum_{s,j,k} |\boldsymbol{f}_j\rangle\langle \boldsymbol{f}_j|\boldsymbol{u}_s\rangle q_s \langle \boldsymbol{u}_s|\boldsymbol{f}_k\rangle\langle \boldsymbol{f}_k| \tag{105}$$

, where I converted to eigenbasis $\boldsymbol{f}_k$ of $\boldsymbol{H}$-matrix. It allows rewriting (104) as

$$q(t) = \sum_{s,j,k} \langle \boldsymbol{u}_0|\boldsymbol{f}_j\rangle\langle \boldsymbol{f}_j|\boldsymbol{u}_s\rangle q_s \langle \boldsymbol{u}_s|\boldsymbol{f}_k\rangle\langle \boldsymbol{f}_k|\boldsymbol{u}_0\rangle \cdot exp\left(i\frac{E_j - E_k}{\hbar}t\right) \tag{106}$$

, where $E_j$ are eigenvalues of $\boldsymbol{H}$. Coefficients $\langle \boldsymbol{u}_0|\boldsymbol{f}_j\rangle\langle \boldsymbol{f}_j|\boldsymbol{u}_s\rangle q_s \langle \boldsymbol{u}_s|\boldsymbol{f}_k\rangle\langle \boldsymbol{f}_k|\boldsymbol{u}_0\rangle$ are all real because their transpose by $j, k$ indices is equal to their adjoint. Hence, I can I rewrite (106) as:

$$q(t) = \sum_{s,j,k} \langle \boldsymbol{u}_0|\boldsymbol{f}_j\rangle\langle \boldsymbol{f}_j|\boldsymbol{u}_s\rangle q_s \langle \boldsymbol{u}_s|\boldsymbol{f}_k\rangle\langle \boldsymbol{f}_k|\boldsymbol{u}_0\rangle \cdot cos\left(\frac{E_j - E_k}{\hbar}t\right) \tag{107}$$

From (107) it is clear that $\left(q(0) - q(t)\right)|_{t \to 0} \propto t^2$ (Misra B., 1977; Koshino K., 2005), which can also be demonstrated if (104) is broken into power series of $t$ near $t = 0$:

$$q(t) = \left\langle \boldsymbol{u}_0 \left| cos\left(ad\left(\frac{\boldsymbol{H}}{\hbar}t\right)\right)\boldsymbol{Q} \right| \boldsymbol{u}_0 \right\rangle = q_0 - \frac{t^2}{2\hbar^2}\langle \boldsymbol{u}_0|[\boldsymbol{H},[\boldsymbol{H},\boldsymbol{Q}]]|\boldsymbol{u}_0\rangle$$

$$+ \frac{t^4}{24\hbar^4}\left\langle \boldsymbol{u}_0 \left| \left[\boldsymbol{H},\left[\boldsymbol{H},[\boldsymbol{H},[\boldsymbol{H},\boldsymbol{Q}]]\right]\right] \right| \boldsymbol{u}_0 \right\rangle + o(t^6) \tag{108}$$

, where $ad(\boldsymbol{H})\boldsymbol{Q} = [\boldsymbol{H},\boldsymbol{Q}] = \boldsymbol{H}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{H}$ are commutator brackets. If the system was in state $\boldsymbol{u}_0$ at $t = 0$ then the probability $P(t)$ to find it in state $\boldsymbol{u}_0$ at $t > 0$ is:

$$P(t) = \sum_{j,k} P_j P_k \cdot cos\left(\frac{E_j - E_k}{\hbar}t\right) \qquad , \text{where} \qquad P_k = |\langle \boldsymbol{f}_k|\boldsymbol{u}_0\rangle|^2 \tag{109}$$

From (107-109) it follows that $\partial q(t)/\partial t \,|_{t=0} = 0$; $\partial P(t)/\partial t \,|_{t=0} = 0$ leading to what is perceived as *quantum Zeno effect*. The eigenvalues $E_j$ of $\mathbf{H}$-matrix are not true energy levels of the system because $\mathbf{H}$-matrix is not a true Hamiltonian, as I noted above. They are also defined up to an arbitrary constant, because only the difference between them matters for the dynamics of the system, according to (107,109).

The meaning of (107) is *expectation value*, not the actual result of the measurement. The measurement always produces one of the eigenvalues of $\mathbf{Q}$. It is a mistake to "confirm" Zeno effect by implying that if the result of the measurement at $t = 0$ is $q_0$ then the result of a subsequent measurement will be given by (107). What (107) provides is the statistical correlation between the initial measurement at $t = 0$ and a subsequent measurement at $t > 0$. As with violation of Bell's inequalities, correlation does not imply causality.

The proposed framework stipulates that the results of multiple measurements is the superposition of knowledge vectors, as in (97), with the second term in (98) providing the statistical correlation between two measurements. Therefore, QM formula (107,109) is a special case of the second term in (98) for the time-separated measurements. The expression for correlation between measurements at $t = 0$ and at $t > 0$ in device eigenspace representation within the proposed framework is:

$$Corr(t,0) = \frac{\sum_i \mathcal{E}_i \cdot cos(\alpha_i - \beta_i)}{\sum \mathcal{E}_i} = \frac{\sum_i \mathcal{E}_i \cdot cos(\omega_i t - \omega_i 0)}{\sum \mathcal{E}_i} = \frac{\sum_i \mathcal{E}_i \cdot cos(\omega_i t)}{\sum \mathcal{E}_i} \qquad (110)$$

And the expression for probability to find the system in original state:

$$P(t) = \left(1 + Corr(t,0)\right)/2 = \frac{1}{2} + \frac{1}{2}\sum_i q_i \cdot cos(\omega_i t) \qquad (111)$$

, where probabilities $q_i = \mathcal{E}_i/\sum \mathcal{E}_i = \mathcal{E}_i/\mathcal{E}$, with $\mathcal{E}_i$ given by (96) and $\omega_i$ introduced in (95). According to (89), for even $M$ at least one $\omega_i$ in (111) is 0. The summation in (111) is by eigenspace of the measuring device. The summation in (109) is by eigenstates of $\mathbf{H}$-matrix. The probabilities of all eigenstates corresponding to the same eigenvalue of $\mathbf{H}$-matrix can be added. Then (109) can be written as a sum over eigenvalues of $\mathbf{H}$-matrix. In a case of two-level system, I connect (109) to (111) as follows:

$$\sum_{j,k} P_j P_k \cdot cos\left(\frac{E_j - E_k}{\hbar}t\right) = P_1^2 + P_2^2 + 2P_1 P_2 cos\left(\frac{E_2 - E_1}{\hbar}t\right) = \frac{1}{2} + \frac{q_1}{2} + \frac{q_2}{2}cos(\omega_2 t) \qquad (112)$$

, hence $q_1 = (P_1 - P_2)^2$ ; $q_2 = 4P_1 P_2$. Thus the classical two-energy-level system corresponds to $M = 4$ -cardinality statistical ensemble.

In a way the presented model bears similarity to *Heisenberg picture*, where evolution in time is represented as a time-parameterized unitary transformation of observation basis, except that in *Heisenberg picture* the wave function, representing the underlying system, is considered static. In my model the observer is the basis from which the system is being observed. Both the observation basis and underlying system are associated with time through relations (1,22,78,91). The time sequence of underlying system is represented by statistical ensembles arranged by time progression rule (2,3). The time dynamics of underlying system in thermodynamic limit is approximated by rather featureless exponential decline (24). The observational diversity is rooted not in a dynamics of underlying system but in a dynamics of observation basis. The equation (92) incorporates both. I propose (92) as a replacement for Schrödinger's equation, for the following reasons:

1. It has the Second Law of Thermodynamics (SLT) built-in, due to -1 terms in diagonal elements of $\boldsymbol{\Phi}$-matrix

2. **Φ**-matrix is diagonal in representation of the measuring device eigenspaces. Therefore (92) incorporates measurement apparatus/observer into the equation.
3. Equations (92) written for measurement devices with different eigenspaces are not unitarily equivalent, in effect reaffirming (Haag, 1955) theorem.
4. According to Schrödinger's equation a closed system (a system in an energy eigenstate) is static, in direct violation of the Second Law of Thermodynamics. A closed system is not static in the presented framework

The Second Law of Thermodynamics (SLT) is the result of time progression of underlying system (Viznyuk, 2011), which is why SLT is not dependent on observation basis. QM is not compliant with SLT, which is evident from the fact that (Von Neumann, 1955) entropy is invariant under unitary transformations, including time propagator $exp(iHt/\hbar)$ transformation.

In preamble I have stated *there is only present*, and what is perceived as the *past* or the *future* are artifacts of the present state. How can that view be reconciled with time dynamics (92) of observables? To better understand the model, take a look at the hand watch. By noticing positions of its short and long hands I immediately know the positions of these hands at a given time in the past, and I readily assume the positions these hands will be at a given time in the future. Thus the present state of the watch incorporates the memory of the *past*, as well as the expectation of the *future*, given rotation frequencies (95). A watch with an hour and minute hands represents a knowledge vector for an underlying system of cardinality $M = 5$. The hour hand represents the lower-energy component (96) of the knowledge vector within one *2-D* eigenspace, and the minute hand represents higher-energy component (96) of the knowledge vector within another *2-D* eigenspace. The underlying system of higher cardinality $M$, and the choice of observables can make time dynamics arbitrary intricate.

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| $N = 1000;\ M = 2;$ $\{p_i\} = \{\frac{1}{2},\frac{1}{2}\}$ | | $N = 900;\ M = 3;$ $\{p_i\} = \{\frac{1}{3},\frac{1}{3},\frac{1}{3}\}$ | | $N = 600;\ M = 5;$ $\{p_i\} = \{\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\}$ | | $N = 189;\ M = 7;$ $\{p_i\} = \{\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7}\}$ | |
|---|---|---|---|---|---|---|---|
| $\mathcal{E}$ | $g(\mathcal{E};N,\{p_i\})$ | $\mathcal{E}$ | $g(\mathcal{E};N,\{p_i\})$ | $\mathcal{E}$ | $g(\mathcal{E};N,\{p_i\})$ | $\mathcal{E}$ | $g(\mathcal{E};N,\{p_i\})$ |
| 0.000000 | 1 | 0.000000 | 1 | 0.000000 | 1 | 0.000000 | 1 |
| 0.001998 | 2 | 0.003328 | 6 | 0.008299 | 20 | 0.036368 | 42 |
| 0.007992 | 2 | 0.009972 | 3 | 0.016598 | 30 | 0.072735 | 210 |
| 0.017982 | 2 | 0.009994 | 3 | 0.024828 | 30 | 0.107827 | 105 |
| 0.031968 | 2 | 0.013311 | 6 | 0.024966 | 30 | 0.109103 | 140 |
| 0.049951 | 2 | 0.023262 | 6 | 0.033127 | 20 | 0.110476 | 105 |
| 0.071930 | 2 | 0.023328 | 6 | 0.033196 | 20 | 0.144194 | 420 |
| 0.097905 | 2 | 0.029951 | 6 | 0.033265 | 20 | 0.145567 | 42 |
| 0.127878 | 2 | 0.039846 | 3 | 0.041495 | 120 | 0.146843 | 420 |
| 0.161847 | 2 | 0.040023 | 3 | 0.049521 | 20 | 0.180562 | 105 |
| 0.199813 | 2 | 0.043196 | 6 | 0.050072 | 20 | 0.181935 | 840 |
| 0.241778 | 2 | 0.043329 | 6 | 0.057889 | 60 | 0.183211 | 105 |
| 0.287740 | 2 | 0.053246 | 6 | 0.058024 | 30 | 0.213187 | 140 |
| 0.337700 | 2 | 0.063064 | 6 | 0.058162 | 30 | 0.215653 | 105 |
| 0.391659 | 2 | 0.063396 | 6 | 0.058302 | 60 | 0.218302 | 1260 |
| 0.449618 | 2 | 0.069775 | 6 | 0.066188 | 60 | 0.220951 | 105 |
| 0.511576 | 2 | 0.069997 | 6 | 0.066393 | 30 | 0.223804 | 140 |
| 0.577534 | 2 | 0.083198 | 6 | 0.066601 | 60 | 0.249555 | 210 |
| 0.647492 | 2 | 0.089556 | 3 | 0.074556 | 60 | 0.250928 | 210 |
| 0.721452 | 2 | 0.090154 | 3 | 0.074696 | 20 | 0.253394 | 630 |
| […] | […] | […] | […] | […] | […] | […] | […] |
| 640.1354 | 2 | 952.5446 | 6 | 929.0220 | 30 | 333.7789 | 105 |
| 644.8379 | 2 | 955.9414 | 3 | 929.4275 | 60 | 334.1844 | 210 |
| 649.6592 | 2 | 956.3468 | 6 | 930.8138 | 20 | 335.5707 | 42 |
| 654.6150 | 2 | 957.7331 | 6 | 934.0276 | 20 | 337.6184 | 140 |
| 659.7260 | 2 | 962.0473 | 6 | 934.7208 | 60 | 338.3115 | 210 |
| 665.0203 | 2 | 963.1459 | 6 | 935.8194 | 20 | 339.4101 | 42 |
| 670.5388 | 2 | 968.1543 | 3 | 940.4212 | 30 | 342.8495 | 105 |
| 676.3459 | 2 | 968.8474 | 6 | 941.1144 | 20 | 343.5426 | 42 |
| 682.5595 | 2 | 974.9556 | 6 | 946.8165 | 20 | 348.0859 | 42 |
| 689.4673 | 2 | 981.7580 | 3 | 953.2134 | 5 | 353.3277 | 7 |

Table 1

$\mathcal{E}$, $g(\mathcal{E};N,\{p_i\})$ value pairs calculated from (9) for four sets of parameters $N,\{p_i\}$ using (Yamanaka, Kawano, & Y., 2007) algorithm for finding partitions $\{n_i\}$ of integer $N$ into $\leq M$ parts. For each partition $\{n_i\}$ I calculated the value of $\mathcal{E}$ and multiplicity $D(\mathcal{E};N,M)$ of multinomial coefficient in (4) (Viznyuk, OEIS sequence A210238, 2012). Finally, $g(\mathcal{E};N,\{p_i\}) = SUM(D)$ for each distinct value of $\mathcal{E}$ produced the results for the table. I display the first 20 and the last 10 records from the table.
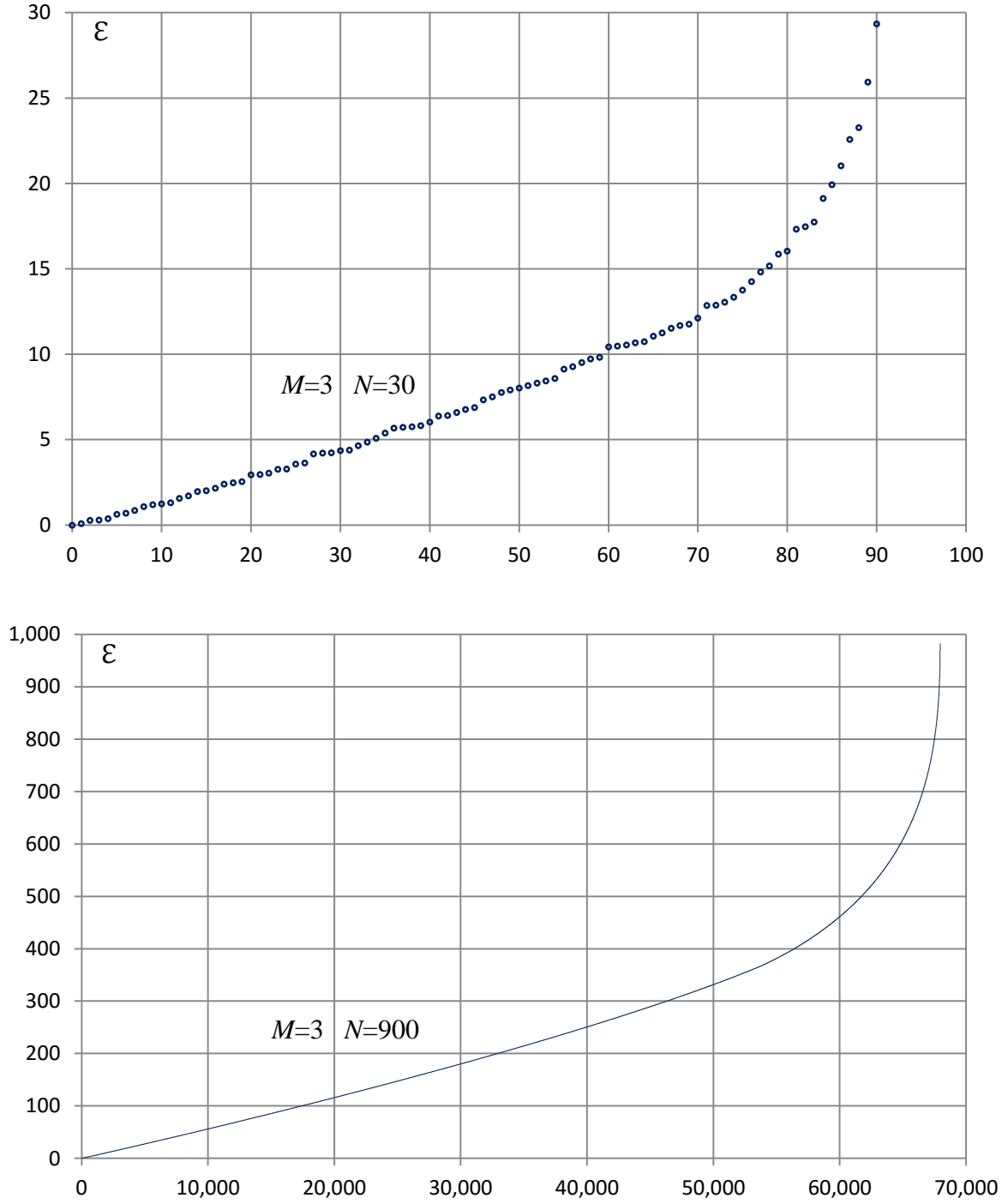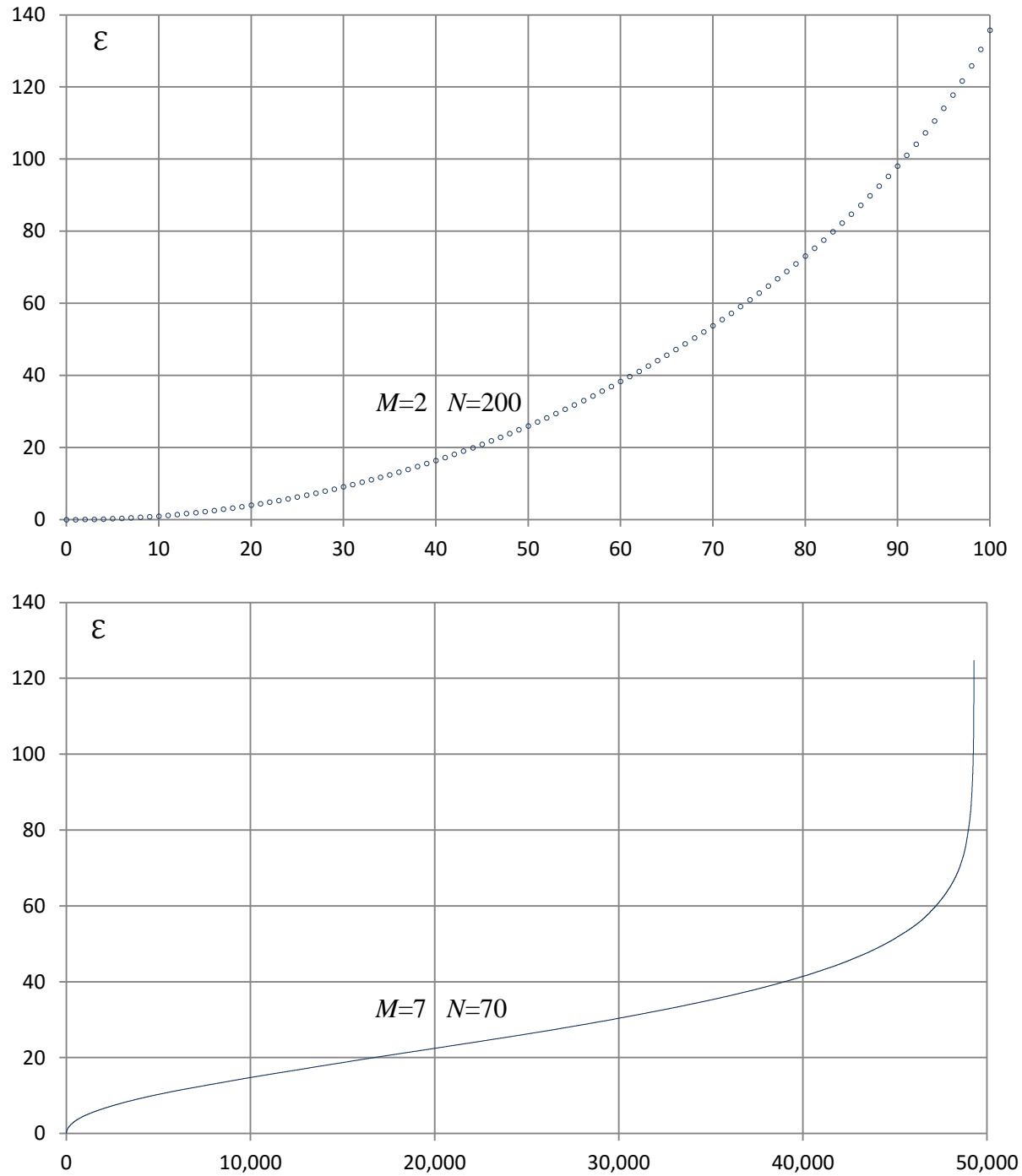
Figure 1

Distinct values of $\mathcal{E}$ in increasing order calculated from (9) with (5) and (1), using (Yamanaka, Kawano, & Y., 2007) algorithm for finding partitions $(n_i)$ of integer $N$ into $\leq M$ parts. The values of $M$ and $N$ are given on the graphs. The graphs represent complete set of distinct values of $\mathcal{E}$ for the given values of $M$ and $N$. The graphs demonstrate close to linear dependence of $\mathcal{E}$ on "quantum number" in the vicinity of equilibrium $\mathcal{E} = 0$ for statistical ensembles with $M=3$. This is the characteristic feature of statistical ensemble of cardinality $M=3$. Away from equilibrium and close to the boundary of hyper-plane (1) the linear behavior is violated.

Figure 2
Distinct values of $\mathcal{E}$ in increasing order calculated from (9) with (5) and (1), using (Yamanaka, Kawano, & Y., 2007) algorithm for finding partitions $(n_i)$ of integer $N$ into $\leq M$ parts. The values of $M$ and $N$ are given on the graphs. The graphs represent complete set of distinct values of $\mathcal{E}$ for the given values of $M$ and $N$.
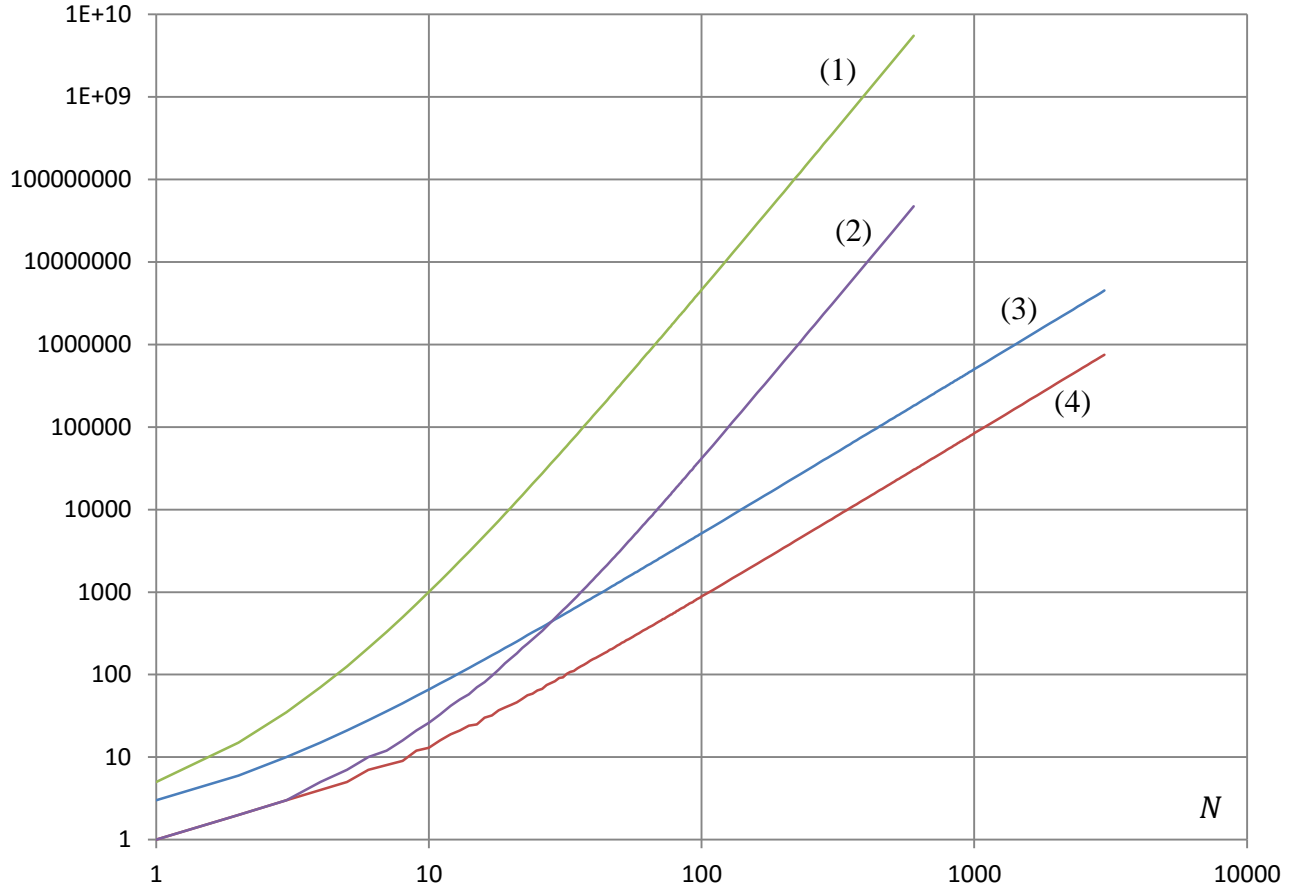
Figure 3

The total number $L(N, M)$ of distinguishable states of statistical ensemble (curves 1, 3), and the total number $\sum_{\{\mathcal{E}\}} 1$ of distinct $\{\mathcal{E}\}$ values (curves 2, 4) as functions of $N$ for two sets of probabilities (5):

1.      $L(N, M)$ for $M = 5$
2.      $\sum_{\{\mathcal{E}\}} 1$ for $M = 5$
3.      $L(N, M)$ for $M = 3$
4.      $\sum_{\{\mathcal{E}\}} 1$ for $M = 3$

The values on curve 1 are by factor $M! = 5!$ greater than on curve 2 as $N \to \infty$. The values on curve 3 are by factor $M! = 3!$ greater than on curve 4 as $N \to \infty$.

Using Stirling's approximation for large $N$ in formula (13) one can see the curves grow proportionally to $N^{M-1}$ as $N \to \infty$
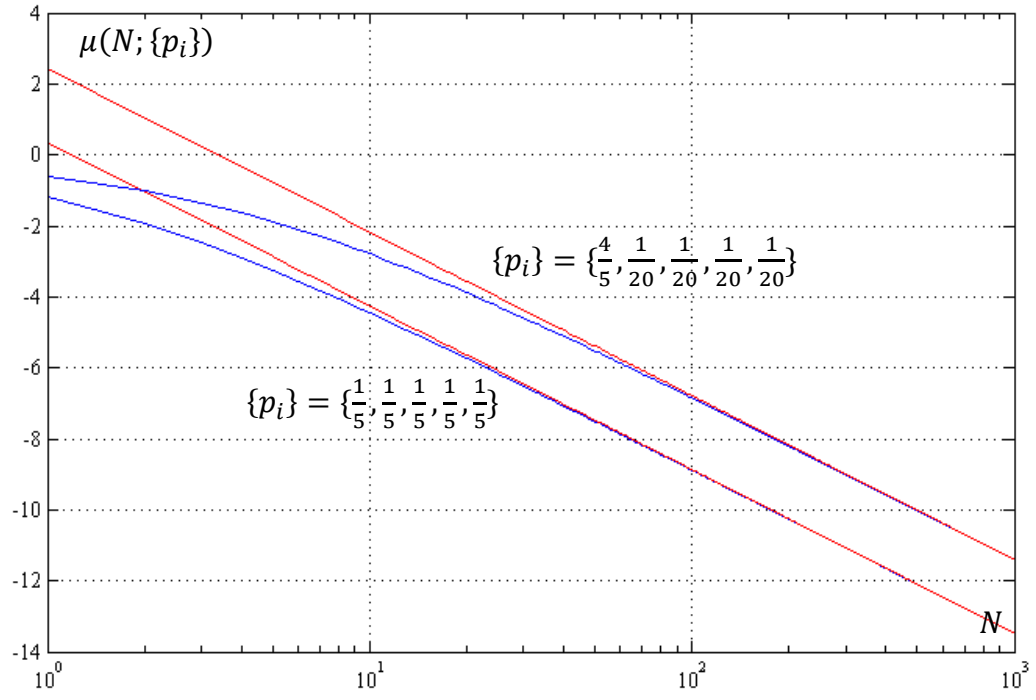
Figure 4

Function $\mu(N; \{p_i\})$ calculated for two sets of probabilities $\{p_i\}$. Blue lines were calculated using exact formula (7). Red lines were calculated using thermodynamic limit approximation (17)
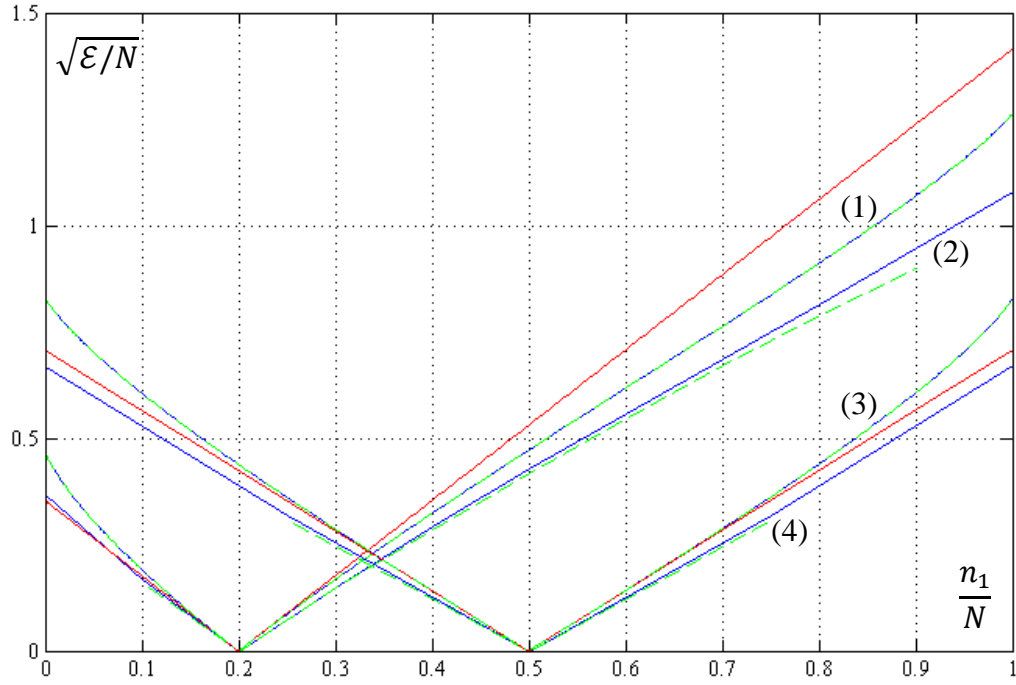
Figure 5

Values of $\sqrt{\mathcal{E}/N}$ calculated as a function of $n_1/N$ with probabilities (5) for four sets of parameters:

1. $M = 5; \ N = 1000$
2. $M = 5; \ N = 10$
3. $M = 2; \ N = 1000$
4. $M = 2; \ N = 4$

Blue lines were calculated using exact formula (9). Green dash lines were calculated using thermodynamic limit approximation (18). Red lines were calculated using quadratic form (23) approximation. For a given value of $n_1$ the values $\{n_{i>1}\}$ were distributed proportionally to corresponding probabilities $\{p_{i>1}\}$. For large value of $N = 1000$ the blue lines and green dash lines overlap closely as seen on curves 1 and 3. For small values of $N$ the thermodynamic limit approximation is not accurate, and blue lines differ from green dash lines as seen on curves 2 and 4. Red lines overlap with blue lines in close proximity to minimum (10) of $\mathcal{E}$.
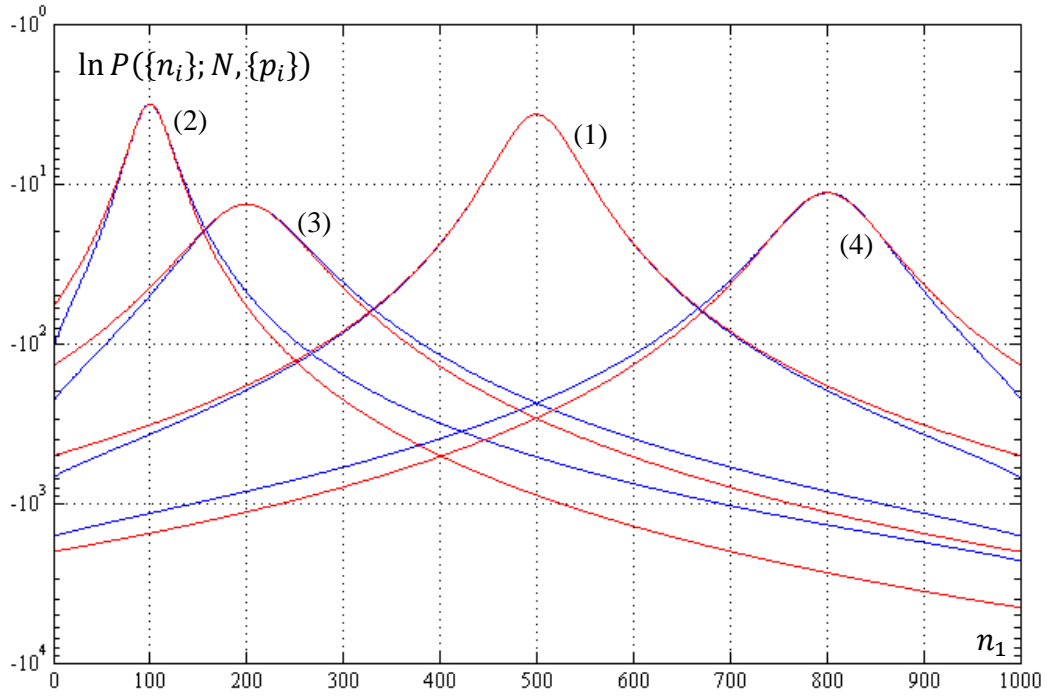
Figure 6

Values of $\ln P(\{n_i\}; N, \{p_i\})$ calculated as a function of $n_1$ for $N = 1000$ and four sets of probabilities $\{p_i\}$

1.  $\{p_i\} = \left\{\frac{1}{2}, \frac{1}{2}\right\}$

2.  $\{p_i\} = \left\{\frac{1}{10}, \frac{9}{10}\right\}$

3.  $\{p_i\} = \left\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right\}$

4.  $\{p_i\} = \left\{\frac{4}{5}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right\}$

Blue lines were calculated using exact formula (4). Red lines were calculated using multivariate normal approximation (25). For the given value of $n_1$ the distribution of values $\{n_{i>1}\}$ is proportional to corresponding probabilities $\{p_{i>1}\}$
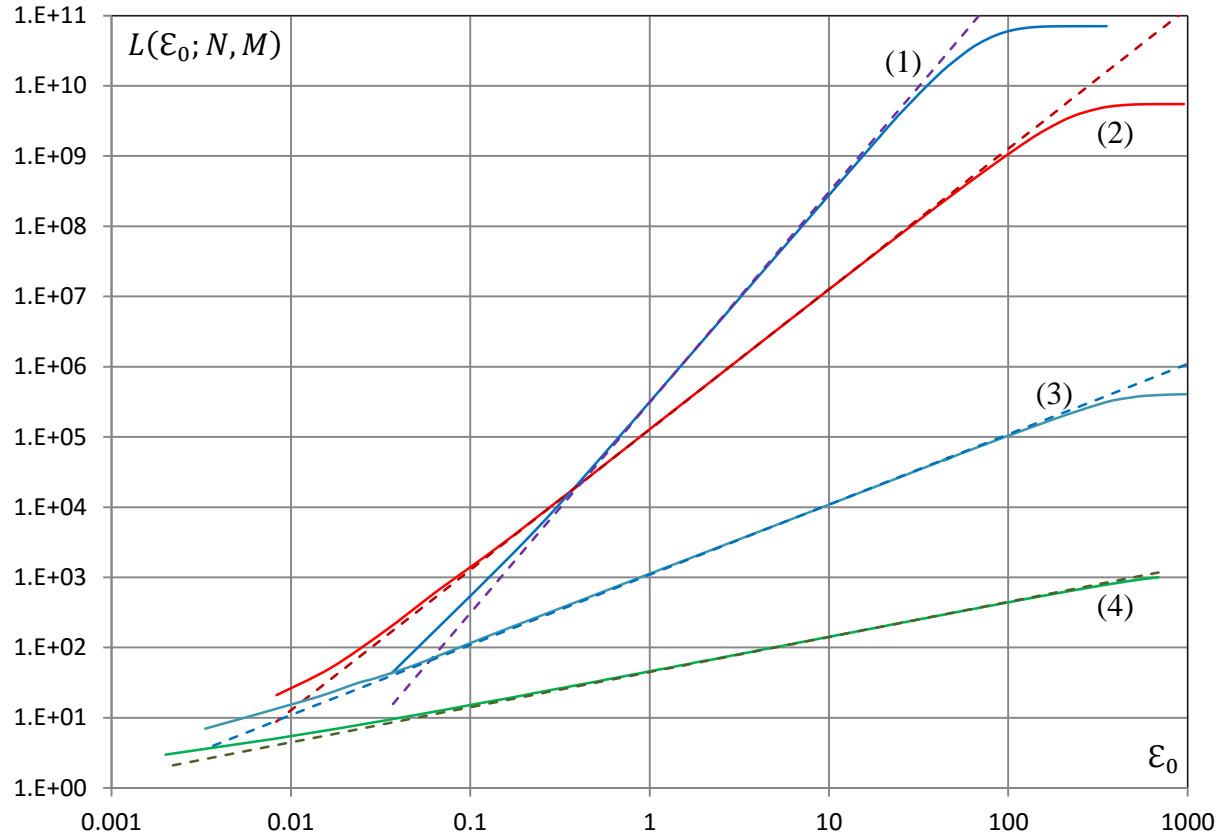
Figure 7

The number of distinguishable states $L(\mathcal{E}_0; N, M)$ of statistical ensemble having $\mathcal{E} \le \mathcal{E}_0$ as a function of $\mathcal{E}_0$ for three sets of the parameters and probabilities (5):

1. $M = 7; N = 189$
2. $M = 5; N = 600$
3. $M = 3; N = 900$
4. $M = 2; N = 1000$

Solid lines are the results of calculation using exact formulas (4) and (9). Dash lines represent thermodynamic limit approximation (34). The graphs demonstrate thermodynamic limit provides the better approximation the larger is the ratio $N/M$. Solid lines level off close to $\mathcal{E}_{max}$ because density of states per interval $d\mathcal{E}$ decreases near $\mathcal{E}_{max}$ due to non-spherical domain $\mathcal{E}$ boundary of hyper-plane (1). The boundary is specified by $n_i \ge 0 \; \forall \; \boldsymbol{i} \in \boldsymbol{G}$
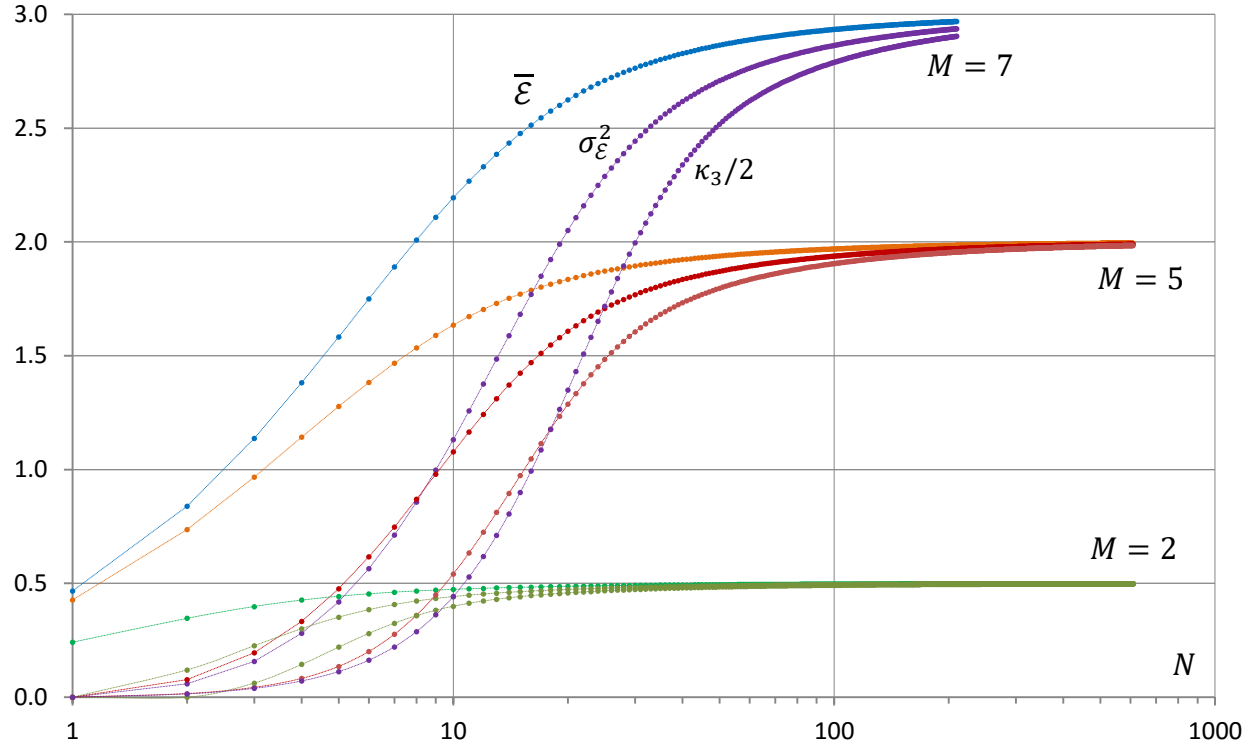
Figure 8

The mean value $\overline{\mathcal{E}}$, the variance $\sigma_{\mathcal{E}}^2$, and the third moment $\kappa_3$ vs. total number $N$ of microstates for three values of $M$ and probabilities (5). The graphs have been calculated using exact expressions (40-42) with probability mass function (4). The value of the third moment $\kappa_3$ is reduced by a factor of 2 to show its asymptotic behavior comparing with $\overline{\mathcal{E}}$ and $\sigma_{\mathcal{E}}^2$. For each set of parameters, the curves approach $(M-1)/2$ values as $N \cdot p_i \to \infty$
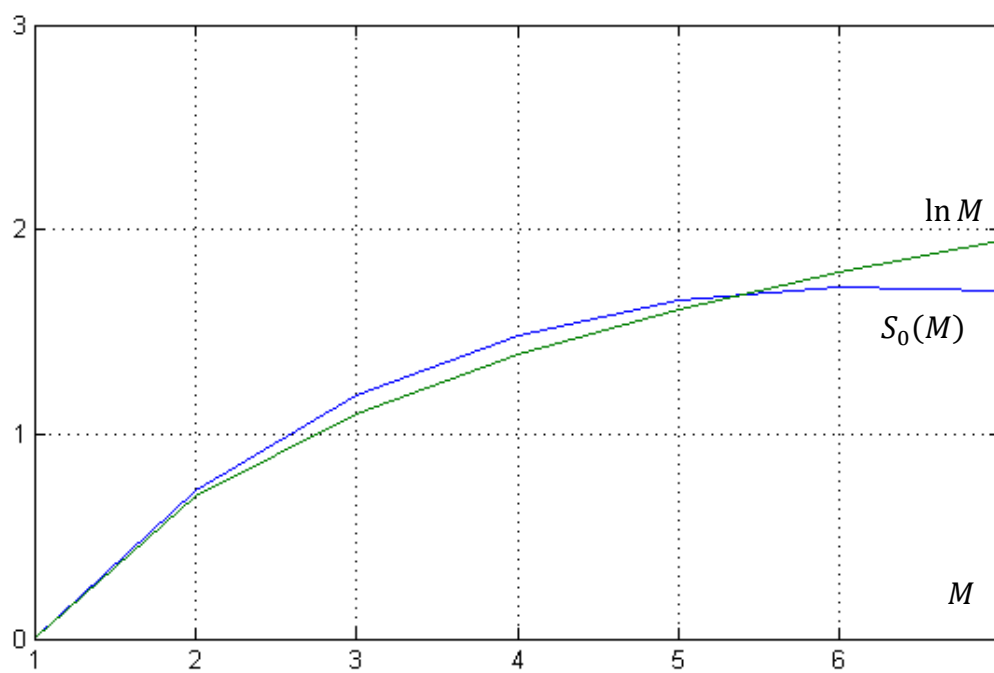
Figure 9
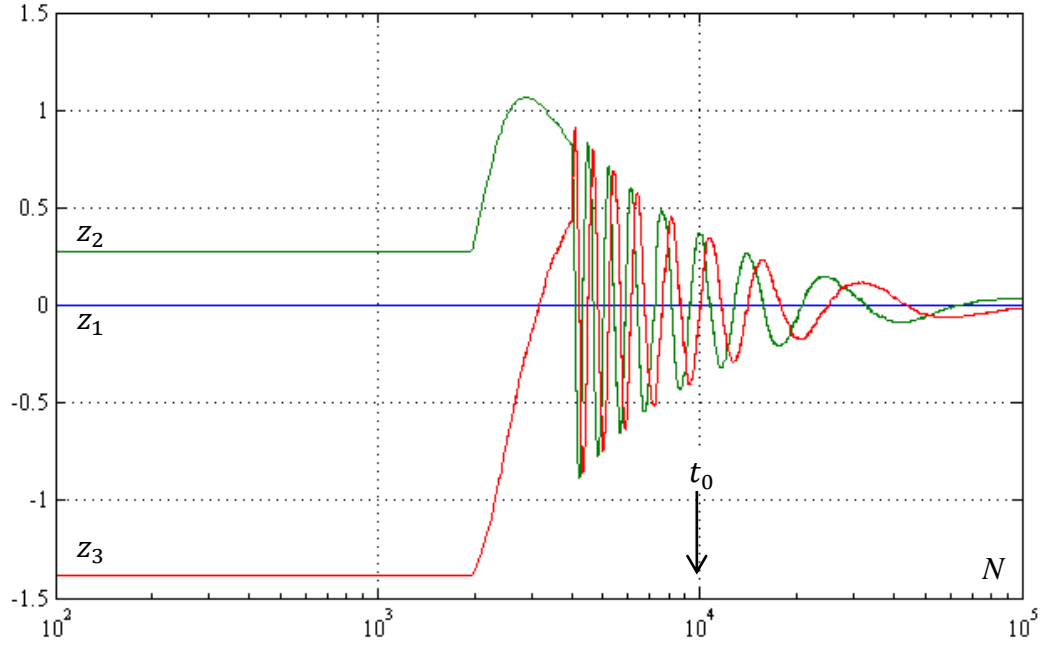Comparison of $S_0(M)$ in (74) with $\ln M$

Figure 10

Time evolution of observables $\{z_i\}$ corresponding to time dynamics of population numbers $\{n_i\}$ on Figure 11. Time dynamics has been calculated for ensemble of microstates with cardinality $M = 3$ using time model (Viznyuk, 2011). Observable values $\{z_i\}$ have been calculated using (78-81) with parameter $W = 50$. The canonical variables $\{x_i\}$ are given by formula (22). The observable $z_1 \equiv 0$ because transformation (78) is restricted to hyper-plane (1), where $x_1 \equiv 0$. Effectively it models one-dimensional damped harmonic oscillator, with variables $z_2$ and $z_3$ representing generalized coordinate and momentum. The corresponding time dynamics of the energy for the system is shown on Figure 12. The proper time of the system is given by $t = \ln(N)$.
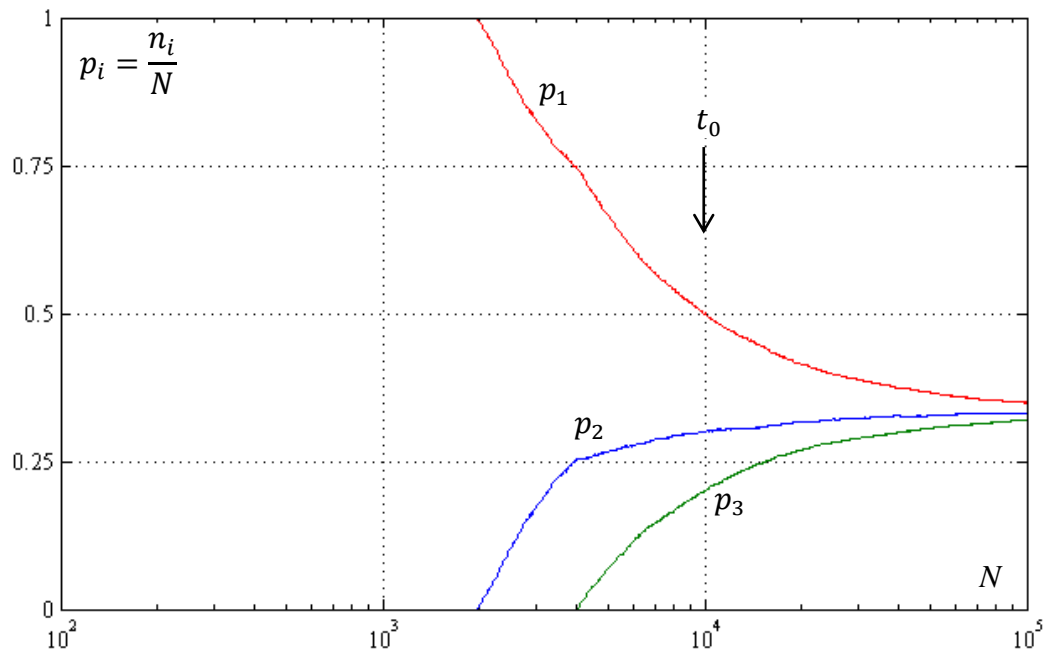
Figure 11

Time dynamics of population numbers $\{n_i\}$ calculated for ensemble of microstates with cardinality $M = 3$ using time model (Viznyuk, 2011). The proper time of the system is given by $t = \ln(N)$. The initial state has been randomly generated for $N = 10^4$, corresponding to proper time $t_0 = \ln(10^4) = 9.21$
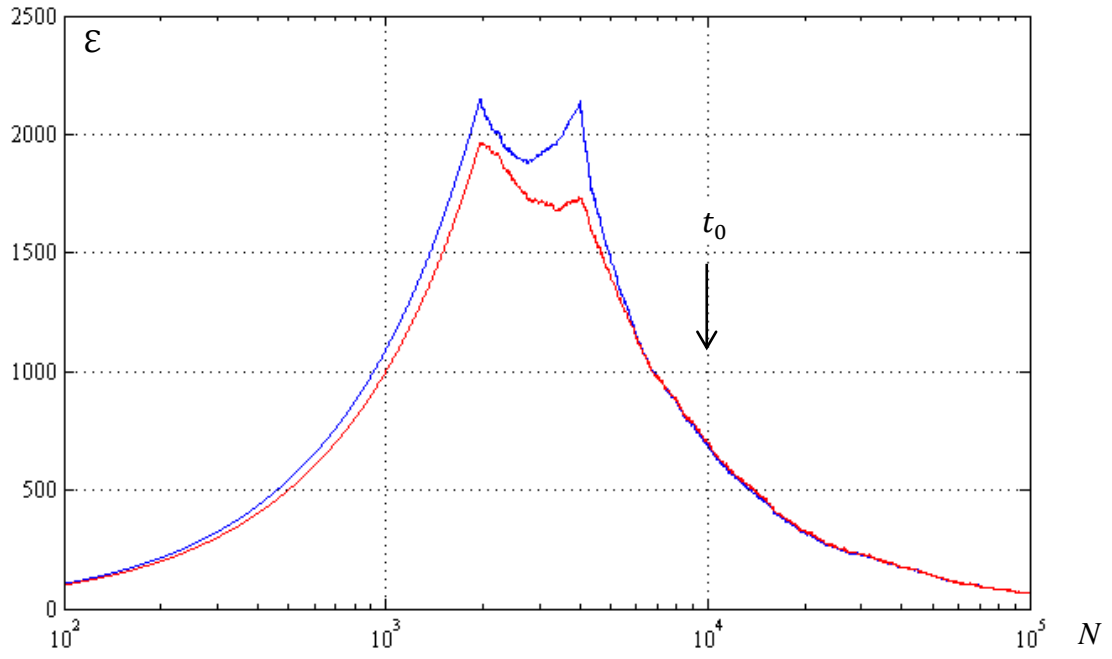
Figure 12

Time dynamics of the energy $\mathcal{E}$ corresponding to evolution of microstate population numbers $\{n_i\}$ on Figure 11. The proper time of the system is given by $t = \ln(N)$. Blue curve has been calculated using exact formula (9). Red curve has been calculated using thermodynamic approximation (23). As population numbers $\{n_i\}$ grow with time the thermodynamic approximation (23) becomes more accurate and red curve and blue curve closely overlap as $(N = \sum n_i) \to \infty$