Planck's law revisited

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Abstract

I review a textbook derivation of Planck's formula for spatial density of radiation energy. I point out at one inconsistency, and a couple of factitious assumptions used in derivation. I propose a derivation more aligned with quantum mechanical principles. I show the de-coherence of oscillator modes is the major factor in Planck's law.

By nearly universal consent, the day of Dec.14, 1900 when Planck's formula [1, 2] for spatial density of radiation energy has been published, is considered [3] the birthday of quantum theory:

$$u_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{\frac{h\nu}{k\vartheta}}-1} \tag{1}$$

Since then, the formula became a staple piece of every textbook on statistical physics [4, 5]. Its close match to experimental results is touted as one of the greatest achievements of quantum theory. For instance, the spectral intensity of cosmic microwave background (CMB) deviates from Planck's formula less than 0.03% [6, 7]. The Planck's derivation [1, 2] of (1) follows from:

$$u_\nu d\nu = \frac{8\pi\nu^2}{c^3}U_\nu d\nu \tag{2}$$

, where U_ν is the average energy of a resonator of frequency ν in a thermodynamic ensemble of resonators, with fixed total energy of the ensemble; $8\pi\nu^2/c^3$ is, supposedly, the number of all hypothetically possible resonator modes per unit volume of space per unit frequency range. The formula for the average energy U_ν is derived from the energy levels of a quantum harmonic oscillator:

$$\mathcal{E}_n = \left(n+\frac{1}{2}\right)\cdot h\nu \tag{3}$$

, and Boltzmann's postulate that probability to find a member of thermodynamic ensemble in a state with energy $\mathcal{E}_n$ is proportional to $exp(-\mathcal{E}_n/kT)$:

$$U_\nu = kT^2\frac{\partial}{\partial T}\ln\sum_{n=0}^{\infty} exp\left(-\frac{\mathcal{E}_n}{kT}\right) = \frac{h\nu}{2} + \frac{h\nu}{exp(h\nu/kT)-1} \tag{4}$$

The reason the so-called *zero-point energy* term $h\nu/2$ in (4) is not included into Planck's formula (1) is the topic of a hundred-year controversy [8, 9]. A short argument follows here. A radiation mode manifests itself via electromagnetic interaction with the matter. As a result of this interaction, the energy quantum $h\nu$ is transferred from the matter to the radiation mode, or from radiation mode to the matter. If radiation mode is in its ground state with $\mathcal{E}_0 = h\nu/2$, it cannot lose another $h\nu$ quantum to the matter. That in itself does not prove the zero-point energy does not exist. However, the gravity which would have been exerted by zero-point energy of all hypothetically possible radiation modes is over 58 orders of magnitude bigger than the gravity which could be derived from empirical evidence [8]. I suggest this discrepancy is because, in textbooks, the radiation is [implicitly] considered a classical object, which exists "out there" in the open space, and possesses properties independent of the measurement context. Such view is contrary to the base principles of quantum mechanics [in Copenhagen interpretation]. I argue that the radiation modes which are present, are the ones which were actually emitted by the matter, not all the hypothetically possible radiation modes. Since the number of oscillators in the matter, which interact with radiation is limited, so is the number of radiation modes. A radiation mode and the corresponding matter oscillator should be viewed as one and the same [entangled] system. Planck may have had similar view, albeit not clearly stated. The word *entanglement* was not in physicist's vocabulary at the time. Planck's reasoning was [2]:

> Let us consider a large number of monochromatically vibrating resonator – N of frequency ν (per second), N' of frequency ν', N'' of frequency ν'', ..., with all N large number – which are at large distances apart and are enclosed in a diathermic medium with light velocity c and bounded by reflecting walls. Let the system contain a certain amount of energy, the total energy E_t (erg) which is present partly in the medium as travelling radiation and partly in the resonators as vibrational energy. The question is how in a stationary state this energy is distributed over the vibrations of the resonator and over the various of the radiation present in the medium, and what will be the temperature of the total system…
> …we first of all consider the vibrations of the resonators and assign to them arbitrary definite energies, for instance, an energy E to the N resonators ν, E' to the N' resonators ν',...
> …Dividing E by N, E' by N',... we obtain the stationary value of the energy U_ν, U'_ν, U''_ν... of a single resonator of each group, and thus also the spatial density of the corresponding radiation energy…

As it sounds, Planck implied the average energy (4) of a resonator in a given resonator group is one and the same as the "corresponding radiation energy". Thus, an actually present radiation mode should have a corresponding matter resonator. The number of radiation modes is the same as the number of matter resonators.

On the other hand, (1) still gives the correct result, even as the factor $8\pi\nu^2/c^3$ is considered as the number of all *hypothetically possible* radiation modes per unit volume of space per unit frequency range, not just actually present modes. This contradiction stems from the way the factor $8\pi\nu^2/c^3$ is derived in textbooks. Note, that (4) is obtained from fundamental quantum mechanical expression (3) in thermodynamic limit of a large number of oscillators present in a given mode. However, the expression $8\pi\nu^2/c^3$ for the number of modes per unit volume per unit frequency range is obtained with purely classical approach, by treating each mode as a standing wave enclosed in a limited volume with ideally conducting walls, so as to nullify the wave amplitude at the boundary. To combine an expression obtained from fundamentally quantum mechanical principles, with an expression obtained from purely classical prospective, into a single formula (1) is the inconsistency I wish to call out. Furthermore, the way the expression $8\pi\nu^2/c^3$ is derived in textbooks [4, 5] ought to raise eyebrows, since it is based on completely improbable assumptions:

1. that the radiation, e.g. cosmic background, can be considered as enclosed in a cavity
2. that the enclosing cavity has ideal conductor walls so the wave amplitude is a mathematical zero at the boundary

Both of these assumptions are crucial for considering the available phase space discrete, which is necessary for the textbook derivation of $8\pi\nu^2/c^3$ factor. Even a miniscule deviation from ideal conductor walls of the cavity immediately breaks the discreteness of phase space, and effectively makes the number of hypothetically possible radiation modes of the same frequency infinite. Thus, the factor $8\pi\nu^2/c^3$ in (2) would have to be derived from a different context, as I do below.

A measurement of radiation intensity with e.g. a bolometer, is effectively the measurement of the energy contained in matter oscillators inside the sensitive element of the device. Consider an oscillator immersed into radiation field. If the system oscillator+radiation is in a state $\boldsymbol{u}_0$ at $t = 0$, the probability to find it in the same state at $t \geq 0$, from Schrödinger equation, is:

$$P(t) = \sum_{j,k} P_j \cdot P_k \cdot cos\left(\frac{E_j - E_k}{\hbar} t\right) \qquad \text{, where } P_k = |\langle \boldsymbol{f}_k | \boldsymbol{u}_0 \rangle|^2 \tag{5}$$

, where $\boldsymbol{f}_k$, E_k are eigenstates and eigenvalues of $\boldsymbol{H}$-matrix. From (5), it follows, $\partial P/\partial t|_{t=0} = 0$, i.e. the transition rate is zero. This result is referred to as quantum Zeno effect [10, 11]. It is the result of a *coherent coupling* (*entanglement*) between the oscillator and radiation modes, manifested by the phase relationship between superposed eigenstates of $\boldsymbol{H}$-matrix in (5):

$$\varphi_j - \varphi_k = \frac{E_j - E_k}{\hbar} t \tag{6}$$

From (5), the transition rate is also zero in a more general case, if phase difference $\varphi_j - \varphi_k$ between $\boldsymbol{f}$-states is *any* analytic function of time, i.e. if phases of $\boldsymbol{f}$-states *predictably relate* to each other. In order for the transition to happen, the superposed eigenstates of $\boldsymbol{H}$-matrix must undergo de-coherence, i.e. the phase relation (6) must be broken. There are various mechanisms which may cause de-coherence of $\boldsymbol{f}$-states, such as:

1. Rayleigh scattering [12, 13]
2. Brownian motion [14, 15]
3. Dispersive media [16, 17]
4. Recombination of electron-hole pairs in semiconductors [18]

It is not in the scope of this paper to consider de-coherence mechanisms in detail. Rather, I shall pursue a generic approach. I write (5) in a more general form, given (6) may no longer hold:

$$P(t) = \sum_{j,k} P_j \cdot P_k \cdot cos(\varphi_j - \varphi_k) \qquad \text{, where } P_k = |\langle \boldsymbol{f}_k | \boldsymbol{u}_0 \rangle|^2 \tag{7}$$

The probability distribution P_k in (7) for an oscillator (e.g. a dipole) over radiation modes $\boldsymbol{f}_k$ is well known (see e.g. dipole radiation). For this discussion it is only important that the number K of radiation modes within a given spectral width $\Delta\nu$ is large, so I can later take the limit $K \to \infty$. If the matter is in equilibrium with radiation, a transition changes oscillator energy by $\Delta\mathcal{E} = \pm h\nu$ with equal probability in either direction. In between transitions, the phase difference $\varphi_j - \varphi_k$ evolves according to (6). The case of $\Delta\mathcal{E} = \pm nh\nu$, where $n > 1$, is equivalent to n consecutive transitions in the same direction. In time t, the phase φ of each of the $\boldsymbol{f}$-states in (7) undergoes a total number t/τ of positive and negative increments with equal probability $1/2$. Here, τ has a meaning of *mean free time* between transitions, i.e. the *de-coherence time*. The resultant increments are binomially distributed around mean $t/(2\tau)$, with variance of binomial distribution $\sigma^2 = p \cdot (1-p) \cdot t/\tau = t/(4\tau)$. The variance in phase $\sigma_\varphi^2 = (\omega\tau)^2\sigma^2 = \omega^2\tau \cdot t/4$. The variance in phase difference is $\sigma_{\Delta\varphi}^2 = 4\sigma_\varphi^2 = \omega^2\tau \cdot t$. Here $\omega = 2\pi\nu$ is the angular frequency.

Figure 1 shows numeric simulation of (7), with binomially distributed phases φ. The calculation established the following:

$$P(t) = \frac{K + K \cdot (K-1) \cdot exp(-\omega^2 \cdot \tau \cdot t)}{K^2} \tag{8}$$

The result (8) is interesting in a couple of ways. First, if $K = 1$, then $P(t) \equiv 1$, i.e. no transition can occur if oscillator couples into a single radiation mode. This result can be obtained directly from (5), since $K = 1$ means the initial state $\boldsymbol{u}_0$ is also the eigenstate of $\boldsymbol{H}$-matrix. Second, (8) shows exponential decay over time, which is the characteristic feature of classical behavior, the result of the de-coherence of radiation modes. In the limit $t \to \infty$, $P(t) \to 1/K$, i.e. the probability spreads equally among radiation modes. That is the consequence of simplification $P_k = 1/K \ \forall \ k$. A more accurate calculation may require a dipole radiation distribution to be used for P_k in (7), a subject for a separate exercise.

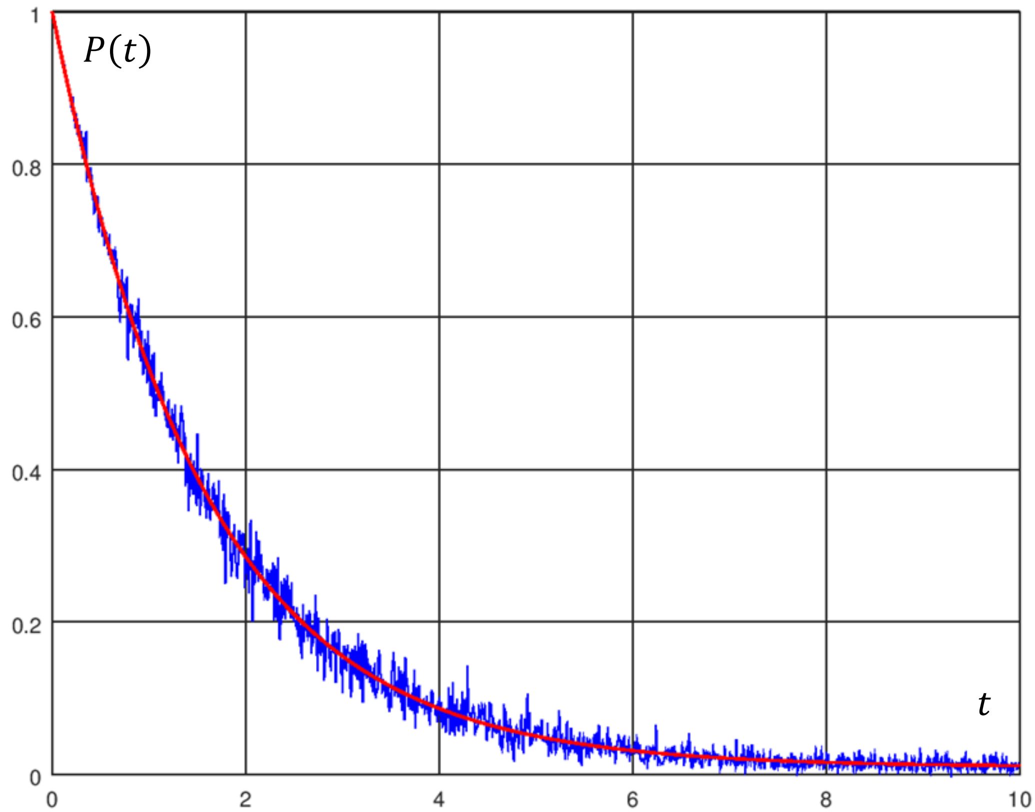

Figure 1

Blue line is a calculation of (7), with binomially distributed phases φ_k. The red line is a plot of formula (8). The graphs were calculated using GNU Octave code http://phystech.com/download/ph2.m with the following parameters:

- $K = 100$
- $\tau = 0.01$
- $\omega = 8$
- $P_k = 1/K \ \forall \ k$

The blue curve smooths out and becomes identical with red curve as $K \to \infty$.

From (8), in the limit of a large number of modes ($K \to \infty$), the transition rate is:

$$\left.\frac{\partial P(t)}{\partial t}\right|_{t=0} = -\omega^2 \cdot \tau \tag{9}$$

To evaluate the de-coherence time τ in (9), I consider a matter oscillator, formed when a set of elements (e.g. electron-hole pairs) on the surface of the detector entangle through some medium (e.g. electromagnetic field). An analog of such entanglement is a Cooper pair in superconductor, mediated by phonon interaction. The surface area which encloses a set of elements in entangled state is limited by the coherence radius $r = 2\pi\, c/(\kappa\Delta\omega)$, where c is the speed of light, κ is the refractive index of the material, and $\Delta\omega$ [*rad/s*] is the spread in internal transition frequencies of the entangled elements. If ρ is the number of entangled elements per unit surface area of the detector; D_ω the dimensionless *scattering rate*; $D_\omega\Delta\omega$ the scattering frequency within $\Delta\omega$ spectral width, then, the de-coherence time τ of the oscillator is:

$$\tau = (\pi r^2 \rho D_\omega \Delta\omega)^{-1} = \left(\pi\left(\frac{2\pi c}{\kappa\Delta\omega}\right)^2 \rho D_\omega \Delta\omega\right)^{-1} = \frac{\kappa^2}{4\pi^3 c^2 \rho D_\omega}\Delta\omega \tag{10}$$

, and the transition rate:

$$\left.\frac{\partial P(t)}{\partial t}\right|_{t=0} = -\frac{\omega^2\kappa^2}{4\pi^3 c^2 \rho D_\omega}\Delta\omega \tag{11}$$

In equilibrium, the loss of a number of oscillators in a particular mode is compensated by the radiation-stimulated induction into the mode of the same number of oscillators. The energy balance equation is

$$R_\omega \cdot B_\omega \cdot \Delta\omega + \xi \cdot \left(U_\omega - \frac{\hbar\omega}{2}\right) \cdot \left.\frac{\partial P(t)}{\partial t}\right|_{t=0} = 0 \tag{12}$$

, where B_ω is the spectral radiance of incident radiation; R_ω is the efficiency of the conversion of the incident radiation into the oscillator energy; ξ is the number of oscillators per unit surface area of the detector. I have to subtract zero-point energy term $\hbar\omega/2$ from the ensemble-average energy U_ω in (12), because an oscillator cannot lose energy in the ground state. Combining (12) with (11):

$$R_\omega \cdot B_\omega = \frac{\xi \cdot \kappa^2}{\rho \cdot D_\omega}\left[\frac{\hbar\omega^3/(4\pi^3c^2)}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1}\right] \tag{13}$$

The term in square brackets can be considered as pertaining to the incident radiation, and parameters outside the brackets as properties of the detector. Then, (13) can be split into formula for the spectral radiance, and formula for the detector efficiency:

$$B_\omega = \frac{\hbar\omega^3/(4\pi^3c^2)}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \tag{14}$$

$$R_\omega = \frac{\xi \cdot \kappa^2}{\rho \cdot D_\omega} = \frac{\kappa^2}{\eta \cdot D_\omega} \tag{15}$$

, where $\eta = \rho/\xi$ can be interpreted as the number of entangled elements making up one oscillator. The Planck's formula (1) for the spectral energy density readily follows from (14):

$$u_\omega = \frac{4\pi}{c} B_\omega = \frac{\hbar\omega^3/(\pi^2c^3)}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \tag{16}$$

$$u_\nu = 2\pi \cdot u_\omega = \frac{8\pi h\nu^3/c^3}{\exp\left(\frac{h\nu}{kT}\right) - 1} \tag{17}$$

I have shown the Planck's law follows from consideration of radiation and matter oscillators as parts of the same quantum system. I have argued against considering radiation as an entity which exists and possesses properties independent of the matter it interacts with.

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