The measurement and the state evaluation

Sergei Viznyuk

Abstract

I critically analyze the fidelity measure used for state estimation. I discuss the impossibility of complete determination. As an alternative to traditional fidelity, I suggest a figure of merit called *confidence* in the knowledge of an arbitrary state

The question of how well one can determine the state of an object by performing measurement is of rather fundamental nature, laying at the base of most scientific disciplines. Here I shall provide a general approach to evaluation of degree of confidence in the knowledge of an arbitrary object, based on the fact that any measurement and associated knowledge is represented by a sample of events (symbols), each symbol being the outcome of a measurement event.

Consider the measurements are done in preparation + measurement cycles (*PMC*). The input to each *PMC* is $\mathbf{a}$, and the output is one of output events $\{\boldsymbol{b}_i\}$, $1 \le i \le d$; where d is the dimension of measurement basis. If *PMC* is repeated N times, the full input is represented by the tensor product $\mathbf{A} = \boldsymbol{a}^{\otimes N}$, and the output by $\mathbf{B} = \boldsymbol{b}_1^{\otimes n_1} \otimes \boldsymbol{b}_2^{\otimes n_2} \dots \otimes \boldsymbol{b}_d^{\otimes n_d}$, where $N = \sum_{i=1}^{d} n_i$. Two questions can be asked:

1. how close is **B** to **A**, or, alternatively, how reliably one can determine **A** from **B**
2. what is the probability of **B** given **A**

A figure of merit, called Uhlmann-Jozsa *fidelity* [1, 2] has been defined to answer question 1:

$$\mathcal{F}_U = \left(tr\left(\sqrt{\sqrt{\boldsymbol{\rho_B}}\boldsymbol{\rho_A}\sqrt{\boldsymbol{\rho_B}}} \right) \right)^2 \tag{1}$$

, where $\boldsymbol{\rho_A}$, $\boldsymbol{\rho_B}$ are density matrices of input **A**, and of output **B**. If **A** or **B** is pure, (1) becomes:

$$\mathcal{F} = tr(\boldsymbol{\rho_B}\boldsymbol{\rho_A}) \tag{2}$$

, which is a case of an expression for the expectation value of an operator **X**:

$$\langle \boldsymbol{X} \rangle = tr(\boldsymbol{X}\boldsymbol{\rho_A}) \tag{3}$$

, with $\boldsymbol{\rho_B}$ in (2) being the probability POVM. Expression (2) is Born rule, postulated [3] to be the answer to question 2. When the same measure is used as the answer to both questions, it leads to some issues I discuss below.

The proposition the fidelity can be used for determination [4] of **A** from **B** lays at the foundation of several technologies, such as quantum state tomography (QST), quantum process tomography (QPT) [5]. Due to non-linearity of (1), its practical use for QST is nearly impossible. Linear inversion of (2), or of alternative fidelity measures [6], is used in all situations, even when both inputs and outputs are mixtures [4]. Even as (1) is touted as a measure of closeness between $\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_B}$, specifically for mixtures, (1) does not make sense from standpoint of closeness of states. For example, if $\boldsymbol{\rho_A} = \boldsymbol{\rho_B} = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ then $\mathcal{F}_U = 1$. However, it should be $1/2$. The reason is, output $\boldsymbol{\rho_B} = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ is a mixture with no correlation to input. Thus, the output $\boldsymbol{b}$ is either $|0\rangle$ or $|1\rangle$ with $1/2$ probability, independent of the input $\boldsymbol{a}$. Hence, $\mathcal{F}_U$ is not the measure of closeness of states, but a measure of closeness of density matrices. For mixtures, density matrix is not synonymous with state but rather with distribution of states. From this prospective, $\mathcal{F}_U = 1$ makes sense, because $\boldsymbol{\rho_A} = \boldsymbol{\rho_B}$. There is an example given in [6] of $\boldsymbol{\rho_B} = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ and $\boldsymbol{\rho_{A=}}|\psi\rangle\langle\psi|$, when (2) gives $\mathcal{F} = 1/2$. In authors' opinion, that is incorrect. However, that is the expected outcome of the measurement. To assume the Uhlmann-Jozsa fidelity (1) provides the figure of merit for closeness of states means accepting possibility of (1) telling **a** is the same as **b**, while measurements show $\boldsymbol{a}$ is different from **b** half the times. I shall conclude, from standpoint of closeness of states, fidelity (2) is the correct measure.

Even as (2) is the correct measure of closeness of states, its use in QST for determination of $\boldsymbol{\rho_A}$ is not faultless, for the following reasons:

1. It is impossible to determine a state in a single-device measurement due to no-cloning theorem [7, 8]. Therefore, (2) implies an ensemble-average. Hence, $\boldsymbol{\rho_A}$ calculated in QST is a mixture, even if the input $\boldsymbol{A}$ is pure. For evidence, the calculated in QST density matrices invariably have multiple non-zero eigenvalues, while pure state density matrix would only have 1 non-zero eigenvalue equal to 1
2. The measure (2) itself cannot be precisely determined in a finite number of measurements, resulting in uncertainty relation formulated below

The optimal state evaluation involves finding a measurement basis $\{\boldsymbol{b}_i\}$ which maximizes (2). From basic geometric consideration it is clear, that in optimal basis [9]:

$$\mathcal{F}_{max}(N_{\boldsymbol{A}}, N_B, d) = \cos^2\left(\varphi_{\boldsymbol{A,B}}^{min}\right) = \frac{d[\boldsymbol{\rho_A}]}{d[\boldsymbol{\rho_B}]} \quad (4)$$

, where $\varphi_{\boldsymbol{A,B}}^{min}$ is the minimum possible Bures [10] angle between **A** and **B**; $d[\boldsymbol{\rho_A}]$ and $d[\boldsymbol{\rho_B}]$ are dimensions [9] of the input and output vector spaces; $d[\boldsymbol{\rho_A}]$ and $d[\boldsymbol{\rho_B}]$ are equal to the number of ways to distribute $N_{\boldsymbol{A}}$ and $N_{\boldsymbol{B}}$ identical balls into d distinguishable cells:

$$d[\rho_{\boldsymbol{A}}] = \binom{N_{\boldsymbol{A}} + d - 1}{N_{\boldsymbol{A}}} \quad ; \quad d[\rho_{\boldsymbol{B}}] = \binom{N_{\boldsymbol{B}} + d - 1}{N_{\boldsymbol{B}}} \quad (5)$$

, where $N_{\boldsymbol{A}}$ is the number of input events, and $N_{\boldsymbol{B}}$ is the number of output events (measurements), $N_{\boldsymbol{B}} \geq N_{\boldsymbol{A}}$. The difference $N_{\boldsymbol{B}} - N_{\boldsymbol{A}}$ is the number of future measurements, given already performed $N_{\boldsymbol{A}}$ *PMC*s. The expression (4) gives the maximum probability that in future $\Delta N = N_{\boldsymbol{B}} - N_{\boldsymbol{A}}$ measurements the result will be the same as in already performed $N_{\boldsymbol{A}}$ measurements. From (4, 5):

$$\mathcal{F}_{max}(N_A; N_{\boldsymbol{B}} = N_A + 1; d) = \frac{N_{\boldsymbol{A}} + 1}{N_{\boldsymbol{A}} + d} \quad (6)$$

$$\mathcal{F}_{max}(N_A \to \infty; N_B \to \infty; d) = \left(\frac{N_{\boldsymbol{A}}}{N_B}\right)^{d-1} \quad (7)$$

$$\mathcal{F}_{max}(N_{\boldsymbol{B}} = 2;\ N_A = 1; d = 2) = \frac{2}{3} \quad (8)$$

Defining $(\Delta\mathcal{F})_{min} = \left(1 - \mathcal{F}_{max}(N_A; N_{\boldsymbol{B}} = N_A + 1; d)\right)$ as the minimum possible uncertainty in state determination, it follows:

$$(\Delta\mathcal{F})_{min} \geq \frac{d - 1}{N_{\boldsymbol{A}} + d} \quad (9)$$

The expression (4), being sensible as probability measure, has an issue from standpoint of fidelity of state determination: the fidelity depends on number of measurements $N_{\boldsymbol{A}}$ already performed, but it cannot depend on the number of future measurements; e.g. (7) does not make sense. This is a conceptual issue of identifying fidelity of state determination with measurement probability (2). In practice [11], the fidelity of optimal state estimation (4) is only used with $N_{\boldsymbol{B}} = N_A + 1$.

To resolve the conceptual issue, I propose an alternative to fidelity measure, which I call the *confidence* in the knowledge of state. The concept of *knowledge* is based on entropy as the measure of missing information. The entropy is the amount of *unknown*. The maximum entropy state, i.e. equilibrium, has zero information content, i.e. zero *known*. Thus, the amount of known, i.e. the *knowledge*, equals the difference between entropy of equilibrium, and the entropy of the estimated state. From here I obtain the expression for *knowledge* [12]:

$$\mathcal{L}\big((n_i); N\big) = H_\Omega^{eq}(N) - H_\Omega\big((n_i);\ N\big) \tag{10}$$

, where

$$H_\Omega\big((n_i);\ N\big) = \ln\Gamma(N+1) - \sum_{1\le i\le d} \ln\Gamma(n_i+1) \tag{11}$$

$H_\Omega\big((n_i);\ N\big)$ is Boltzmann's entropy; $H_\Omega^{eq}(N) = H_\Omega\big((n_i = N/d);\ N\big)$ is entropy of equilibrium. The knowledge obtained per measurement event is:

$$\mathcal{E}\big((n_i); N\big) = \mathcal{L}\big((n_i); N\big)/N \tag{12}$$

Knowledge (12) has its maximum for the given N when $n_j = N;\ n_i = 0\ \forall\ i \ne j$:

$$\mathcal{E}_{max}(N) = \mathcal{E}\left(\big(n_j = N;\ n_i = 0\ \forall\ i \ne j\big); N\right) = \frac{H_\Omega^{eq}(N)}{N} \tag{13}$$

$\mathcal{E}_{max}(N)$ grows with number of measurements N, toward limit:

$$\mathcal{E}_{max} = \mathcal{E}_{max}(N \to \infty) = \ln d \tag{14}$$

As expected, $\mathcal{E}_{max}$ equals maximum per-event entropy, i.e. maximum Shannon's H_S entropy [13]. Once equipped with the notion of knowledge, I define the notion of *confidence* [12] as:

$$\Lambda\big((n_i); N\big) = \frac{\mathcal{E}\big((n_i); N\big)}{\mathcal{E}_{max}} = \frac{\mathcal{E}\big((n_i); N\big)}{\ln d} \tag{15}$$

The fidelity measure (4) for optimal state estimation corresponds to maximum confidence Λ_{max}:

$$\Lambda_{max}(N) = \Lambda\left(\big(n_j = N;\ n_i = 0\ \forall\ i \ne j\big); N\right) = \frac{\mathcal{E}_{max}(N)}{\mathcal{E}_{max}} = \frac{\mathcal{E}_{max}(N)}{\ln d} \\ 0 \le \Lambda_{max}(N) < 1 \tag{16}$$

To summarize, the fidelity (2) is the probability of measurement outcome **B** given input ***A***. The fidelity (4) is the probability of measurement outcome **B** given input ***A*** in optimal state estimation. The *knowledge* (12) is the obtained information (in *nats*) about estimated state, per measurement event. The *confidence* (15) is the ratio of information obtained per measurement event to the maximum possible information per event, which could have been obtained under optimal state estimation with infinite number of measurements.

I shall compare the confidence (16) to fidelity of optimal state estimation (4). Re-normalizing (4) to the same [0,1) domain as confidence, I obtain:

$$\mathcal{F}'_{max}(N_{\boldsymbol{A}}, N_B, d) = \frac{\mathcal{F}_{max}(N_{\boldsymbol{A}}, N_B, d) - \mathcal{F}_{max}(N_{\boldsymbol{A}} = 0, N_B, d)}{1 - \mathcal{F}_{max}(N_{\boldsymbol{A}} = 0, N_B, d)} \tag{17}$$

The calculation of re-normalized fidelity (17) and confidence (16) vs number of input events ($N = N_A$) is presented on Figure 1, for varying $\Delta N = N_B - N_A$; and $d = 4$ dimension of the measurement basis. The figure demonstrates the confidence (16) is close to fidelity (6) of optimal state estimation, i.e. when $\Delta N = 1$. It also shows (4) loses its meaning of fidelity of optimal state estimation when $\Delta N > 1$. I conclude the *confidence* (15, 16) provides the correct figure of merit for state estimation.

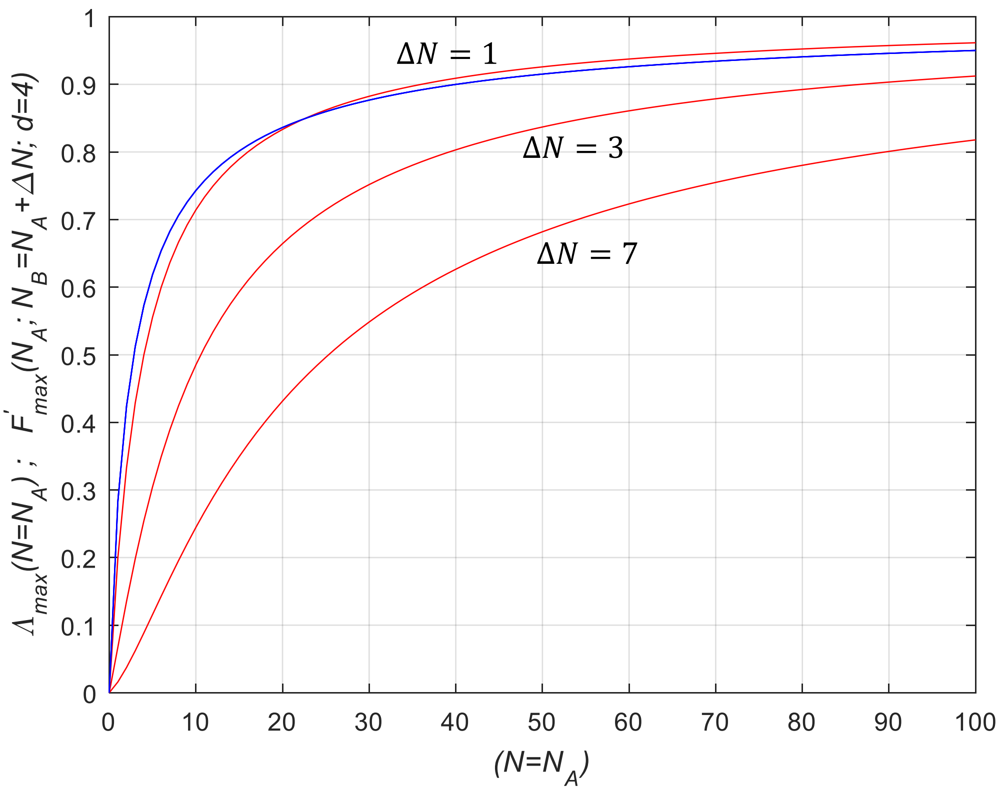

Figure 1

Graphs of confidence (16) and fidelity (17) vs number of measurements N.

Blue line: confidence (16).
Red lines: re-normalized fidelity (17) for several values of $\Delta N = N_B - N_A$.
The calculation was done for dimension of measurement basis $d = 4$.
The MATLAB code used for calculation:
http://www.phystech.com/download/fidelity.m

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